Clifford Algebras and Euclid's Parameterization of Pythagorean Triples

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Abstract. We show that the space of Euclid's parameters for Pythagorean triples is endowed with a natural symplectic structure and that it emerges as a spinor space of the Clifford algebra $\mathbb{R}_{21}$, whose minimal version may be conceptualized as a 4-dimensional real algebra of "kwaternions." We observe that this makes Euclid's parameterization the earliest appearance of the concept of spinors. We present an analogue of the "magic correspondence" for the spinor representation of Minkowski space and show how the Hall matrices fit into the scheme. The latter obtain an interesting and perhaps unexpected geometric meaning as certain symmetries of an Apollonian gasket. An extension to more variables is proposed and explicit formulae for generating all Pythagorean quadruples, hexads, and decuples are provided.

Keywords: Pythagorean triples, Euclid's parameterization, spinors, Clifford algebra, Minkowski space, pseudo-quaternions, modular group, Hall matrices, Apolloniam gasket, Lorentz group, Pythagorean quadruples and n-tuples.

1. Introduction

In the common perception, Clifford algebras belong to modern mathematics. It may then be a point of surprise that the first appearance of spinors in mathematical literature is already more than two thousand years old! Euclid's parameters m and n of the Pythagorean triples, first described by Euclid in his *Elements*, deserve the name of Pythagorean spinors — as will be shown.

The existence of Pythagorean triples, that is triples of natural numbers (a, b, c) satisfying

$$a^2 + b^2 = c^2 \tag{1.1}$$

has been known for thousands of years[1]. Euclid (ca 300 BCE) provided a formula for finding Pythagorean triples from any two positive integers m and n, $m > n$,

[1]The cuneiform tablet known as Plimpton 322 from Mesopotamia enlists 15 Pythagorean triples and is dated for almost 2000 BCE. The second pyramid of Giza is based on the 3-4-5 triangle quite perfectly and was build before 2500 BCE. It has also been argued that many megalithic constructions include Pythagorean triples [17].

namely:

$$\begin{aligned} a &= m^2 - n^2 \\ b &= 2mn \\ c &= m^2 + n^2 \end{aligned} \tag{1.2}$$

(Lemma 1 of Book X). It easy to check that a, b and c so defined automatically satisfy Eq. (1.1), i.e., they form an integer right triangle.

A Pythagorean triple is called *primitive* if (a, b, c) are mutually prime, that is $\gcd(a, b, c) = 1$. Every Pythagorean triple is a multiple of some primitive triples. The primitive triples are in one-to-one correspondence with relatively prime pairs (m, n), $\gcd(m, n) = 1$, $m > n$, such that exactly one of (m, n) is even [15, 16].

An indication that the pair (m, n) forms a *spinor* description of Pythagorean triples comes from the well-known fact that (1.2) may be viewed as the square of an integer complex number. After recalling this in the next section, we build a more profound analysis based on a 1:2 correspondence of the integer subgroups of $O(2, 1)$ and $SL^{\pm}(2, \mathbb{R})$, from which the latter may be viewed as the corresponding pin group of the former. For that purpose we shall also introduce the concept of pseudo-quaternions ("kwaternions") — representing the minimal Clifford algebra for the pseudo-Euclidean space $\mathbb{R}^{2,1}$.

The most surprising result concerns an Apollonian gasket where all the objects mentioned acquire a geometric interpretation.

2. Euclid's Labels as Complex Numbers

Squaring an integer complex number $z = m+ni$ will result in an integer complex number

$$z^2 = (m + n\,i)^2 = (m^2 - n^2) + 2mn\,i, \tag{2.1}$$

the norm of which turns out to be also an integer:

$$|z^2| \;=\; m^2 + n^2\,. \tag{2.2}$$

This auspicious property gives a method of producing Pythagorean triangles: just square any integer complex number and draw the result. For instance $z = 2+i$ will produce 3-4-5 triangle, the so-called Egyptian triangle (Figure 2b). Equation 2.1 is equivalent to Euclid's formula for the parameterization of Pythagorean triples [16]. We shall however allow z to be any integer complex number, and therefore admit triangles with negative legs (but not hypotenuses). The squaring map

$$\text{sq: } \mathbb{C} \to \mathbb{C} : z \to z^2$$

(exclude zero) has a certain redundancy — both z and $-z$ give the same Pythagorean triple. This double degeneracy has the obvious explanation: squaring a complex number has a geometric interpretation of doubling the angle. Therefore one turn of the parameter vector $z = m + ni$ around the origin makes the Pythagorean vector z^2 go twice around the origin. We know the analogue of such a situation

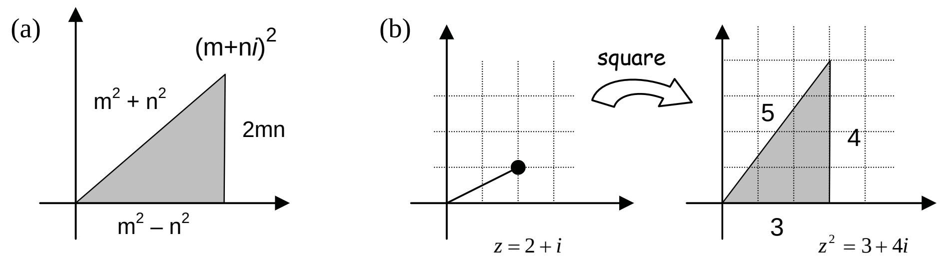

FIGURE 1. (a) Euclid's parameterization. (b) The Egyptian triangle is the square of $z = 2 + i$.

from quantum physics. Fermions need to be turned in the visual space twice before they return to the original quantum state (a single rotation changes the phase of the "wave function" by 180°). Mathematically this corresponds to the group homomorphism of a double cover

$$SU(2) \xrightarrow{2:1} SO(3)$$

which is effectively exploited in theoretical physics [2]. The rotation group $SO(3)$ is usually represented as the group of special orthogonal 3×3 real matrices, and the unitary group $SU(2)$ is typically realized as the group of special unitary 2×2 complex matrices. The first acts on $\mathbb{R}^3$ — identified with the visual physical space, and the latter acts on $\mathbb{C}^2$ — interpreted as the spin representation of states. One needs to rotate a visual vector twice to achieve a single rotation in the spinor space $\mathbb{C}^2$. This double degeneracy of rotations can be observed on the quantum level [1], [14].

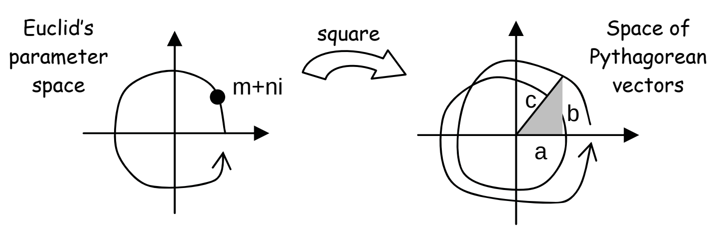

FIGURE 2. Double degeneracy of the Euclid's parameterization.

Remark 2.1. The group of rotations $SO(3)$ has the topology of a 3-dimensional projective space, and as such has first fundamental group isomorphic to $\mathbb{Z}_2$. Thus it admits loops that cannot be contracted to a point, but a double of a loop (going twice around a topological "hole") becomes contractible. Based on this fact, P.A.M. Dirac designed a way to visualize the property as the so-called "belt trick" [5, 6, 12].

This analogy between the relation of the Pythagorean triples to the Euclid's parameters and the rotations to spinors has a deeper level — explored in the following sections, in which we shall show that (2.1) is a shadow of the double covering homomorphism $SL^{\pm}(2) \to O(2,1)$. It shall become clear that calling the Euclid parameterization (m,n) a *Pythagorean spinor* is legitimate.

Before we go on, note another interesting redundancy of the Euclidean scheme. For a complex number $z = m + ni$ define

$$z^d = (m+n) + (m-n)i\,.$$

The square of this number brings

$$(z^d)^2 = 4mn + 2(m^2 - n^2)\; i \tag{2.3}$$

with the norm $|z^d|^2 = 2(m^2+n^2)$. Thus both numbers z and z^d give – up to scale – the same Pythagorean triangle but with the legs interchanged. For instance both $z = 2+i$ and $z^d = 3+i$ give Pythagorean triangles 3-4-5 and 8-6-10, respectively (both similar to the Egyptian triangle). In Euclid's recipe (1.2), only one of the two triangles is listed. This map has "near-duality" property:

$$(z^d)^d = 2z\,. \tag{2.4}$$

3. Pseudo-quaternions and Matrices

A. Pseudo-quaternions. Let us introduce here an algebra very similar to the algebra of quaternions.

Definition. Pseudo-quaternions $\mathbb{K}$ (kwaternions) are numbers of type

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \tag{3.1}$$

where $a, b, c, d \in \mathbb{R}$ and where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are independent "imaginary units". Addition in $\mathbb{K}$ is defined the usual way. Multiplication is determined by the following rules for the "imaginary units":

$$\begin{aligned} \mathbf{i}^2 &= 1 & \mathbf{ij} &= \mathbf{k} \\ \mathbf{j}^2 &= 1 & \mathbf{jk} &= -\mathbf{i} \\ \mathbf{k}^2 &= -1 & \mathbf{ki} &= -\mathbf{j} \end{aligned} \tag{3.2}$$

plus the anticommutation rules for any pair of distinct imaginary units: $\mathbf{ij} = -\mathbf{ji}$, $\mathbf{jk} = -\mathbf{kj}$, and $\mathbf{ki} = -\mathbf{ik}$. [The rules are easy to remember: the minus sign appears only when $\mathbf{k}$ is involved in the product].

The pseudo-quaternions form an associative, non-commutative, real 4-dimensional algebra with a unit. Define the *conjugation* of q as

$$\overline{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} \tag{3.3}$$

and the squared norm

$$|q|^2 = \overline{q}q = a^2 - b^2 - c^2 + d^2 \tag{3.4}$$

The norm is not positive definite, yet kwaternions "almost" form a division algebra. Namely the inverse

$$q^{-1} = \overline{q}/|q|^2 \tag{3.5}$$

is well-defined except for a subset of measure zero — the "cone" $a^2-b^2-c^2+d^2=0$.

Other properties like $|ab|^2 = |ba|^2$, $|ab|^2 = |a|^2|b|^2$, etc., are easy to prove. The set $G_o = \{q \in \mathbb{K} \mid |q|^2 = 1\}$ forms a group, and $G_o \cong SL(2,\mathbb{R})$. Topologically, the group (subset of kwaternions) is a product of the circle and a plane, $G \cong S^1 \times \mathbb{R}^2$. The bigger set $G = \{q \in \mathbb{K} \mid |q|^2 = \pm 1\}$ is isomorphic to $S^{\pm}L(2,R)$, the modular group of matrices with determinant equal ± 1.

The regular quaternions describe rotations of $\mathbb{R}^3$. Pseudo-quaternions describe Lorentz transformations of Minkowski space $\mathbb{R}^{2,1}$, $O(2,1)$. Indeed, note that the map

$$v \to v' = qvq^{-1} \tag{3.6}$$

preserves the norm, $|qvq^{-1}|^2 = |v|^2$. Thus the idea is quite similar to regular quaternions. Represent the vectors of space-time $\mathbb{R}^{2,1}$ by the imaginary part

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + t\mathbf{k} \tag{3.7}$$

Its norm squared is that of Minkowski space $\mathbb{R}^{2,1}$, $|\mathbf{v}|^2 = -x^2 - y^2 + z^2$. Define a *transformation kwaternion* as an element $q \in G$. Special cases are:

$$\begin{array}{rll} \text{rotation in } xy\text{-plane:} & q = \cos\phi/2 + \mathbf{k}\sin\phi/2 & \\ \text{hyperbolic rotations} & q = \cosh\phi/2 + \mathbf{i}\sinh\phi/2 & \text{(boost in direction of } y\text{-axis)} \\ (\textit{boosts}): & q = \cosh\phi/2 + \mathbf{j}\sinh\phi/2 & \text{(boost in direction of } x\text{-axis).} \end{array} \tag{3.8}$$

It is easy to see that the norm of $\mathbf{v}$ is preserved transformation (3.7), as well as its lack of real part.

B. Pseudo-quaternions as a Clifford algebra. The algebra of pseudo-quaternions is an example of a Clifford algebra. Namely, it is the Clifford algebra of a pseudo-Euclidean space $\mathbb{R}^{2,1}$, a 3-dimensional Minkowski space with quadratic form

$$g = -x^2 - y^2 + z^2 \tag{3.9}$$

The corresponding scalar product of two vectors will be denoted as $g(v,w) = \langle v, w\rangle$. Let $\{\mathbf{f}_1,\, \mathbf{f}_2,\, \mathbf{f}_3\}$ be an orthonormal basis of $\mathbb{R}^{2,1}$ such that:

$$\begin{aligned} \langle \mathbf{f}_1, \mathbf{f}_1\rangle &= -1 \\ \langle \mathbf{f}_2, \mathbf{f}_2\rangle &= -1 \\ \langle \mathbf{f}_3, \mathbf{f}_3\rangle &= 1 \\ \text{and } \langle \mathbf{f}_i, \mathbf{f}_j\rangle &= 0 \text{ for any } i \neq j\,. \end{aligned} \tag{3.10}$$

Let us build the Clifford algebra [13] of this space. We shall assume the standard sign convention:

$$\mathbf{vw} + \mathbf{wv} = -2\langle \mathbf{v}, \mathbf{w}\rangle$$

(Clifford products on the left side, pseudo-Euclidean scalar product on the right). The basis of the universal Clifford algebra $\mathbb{R}_{2,1}$ of the space $\mathbb{R}^{2,1}$ consists of eight elements

$$\mathbb{R}_{2,1} = \text{span}\{\, 1,\ \mathbf{f}_1,\ \mathbf{f}_2,\ \mathbf{f}_3,\ \mathbf{f}_1\mathbf{f}_2,\ \mathbf{f}_2\mathbf{f}_3,\ \mathbf{f}_3\mathbf{f}_1,\ \mathbf{f}_1\mathbf{f}_2\mathbf{f}_3\,\}$$

The basic elements satisfy these relations (in the sense of the *algebra* products (cf. (3.2)) :

$$\begin{aligned} \mathbf{f}_1^2 &= 1 \\ \mathbf{f}_2^2 &= 1 \\ \mathbf{f}_3^2 &= -1 \\ \text{and } \mathbf{f}_i\mathbf{f}_j &= -\mathbf{f}_j\mathbf{f}_i \text{ for any } i \neq j\,. \end{aligned} \tag{3.11}$$

Relations between the other elements of the basis of the Clifford algebra are induced from these via the associativity of the algebra product.

C. Matrix representation of pseudo-quaternions. Note that the following four matrices satisfy the above relations (3.11):

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.12}$$

Indeed, the squares are, respectively:

$$\sigma_0^2 = \sigma_0 \qquad \sigma_1^2 = \sigma_0 \qquad \sigma_2^2 = \sigma_0 \qquad \sigma_3^2 = -\sigma_0\,. \tag{3.13}$$

The products are

$$\sigma_1\sigma_2 = +\sigma_3 \qquad \sigma_2\sigma_3 = -\sigma_1 \qquad \sigma_3\sigma_1 = -\sigma_2 \tag{3.14}$$

and $\sigma_0\sigma_i = \sigma_i$ for each i. Thus these matrices play a similar role as the Pauli matrices in the case of spin description or the regular quaternions. Note however the difference: the above construct is over real numbers while the Pauli matrices are over complex numbers.

The other elements are represented by matrices as follows:

$$\mathbf{1} = \sigma_0 \qquad \mathbf{f}_1\mathbf{f}_2 = \sigma_3 \qquad \mathbf{f}_2\mathbf{f}_3 = -\sigma_1 \qquad \mathbf{f}_3\mathbf{f}_1 = -\sigma_2 \qquad \mathbf{f}_1\mathbf{f}_2\mathbf{f}_3 = -\sigma_0\,.$$

As in the case of quaternions, the *universal* Clifford algebra for $\mathbb{R}^{2,1}$ is 8-dimensional, yet the *minimal* Clifford algebra is of dimension 4 and is isomorphic to the algebra of kwaternions $\mathbb{K}$.

Remark 3.1. In algebra, pseudo-quaternions are also known as split quaternions, especially in the context of the CayleyDickson construction [3]. Other names include para-quaternions, coquaternions and antiquaternions.

4. Minkowski Space of Triangles and Pythagorean Spinors

Now we shall explore the geometry of Euclid's map for Pythagorean triples. As a map from a 2-dimensional symplectic space to a 3-dimensional Minkowski space, we have:

$$\begin{aligned} \varphi : (\mathbb{R}^2, \omega) &\to (\mathbb{R}^{2,1}, G) \\ (m, n) &\to (m^2 - n^2, 2mn,\ m^2 + n^2) \end{aligned} \tag{4.1}$$

The map (2.1) of squaring complex numbers is just its truncated version. We shall discuss symmetries of the natural structures of both spaces. Obviously, we are mostly interested in the discrete subsets $\mathbb{Z}^2$ and $\mathbb{Z}^3$ of those spaces.

First, introduce the *space of triangles* (x, y, z) as a real 3-dimensional Minkowski space $\mathbb{R}^{2,1}$ with a quadratic form

$$Q = -x^2 - y^2 + z^2 \tag{4.2}$$

("space-time" with the hypotenuse as the "time"), and with metric given by a 3×3 matrix $G = diag(1, 1, -1)$. The right triangles are represented by "light-like" (null) vectors, and the Pythagorean triples by the integer null vectors in the light cone. Clearly, not all vectors correspond to real triangles, and the legs may assume both positive and negative values. We have immediately

Proposition 4.1. *The group of integer orthogonal matrices $O(2, 1; \mathbb{Z}) \subset O(2, 1; \mathbb{R})$ (Lorentz transformations) permutes the set of Pythagorean triangles.*

On the other hand we have the two-dimensional space of Euclid's parameters $\mathbf{E} \cong \mathbb{R}^2$. Its elements will be called *Pythagorean spinors.* Occasionally we shall use the isomorphism $\mathbb{R}^2 \cong \mathbb{C}$ and identify $[m, n]^T = m + ni$.

The space of Euclid's parameters will be equipped with an inner product:

Definition 4.2. For two vectors $u = [m, n]^T$ and $w = [m', n']^T$, the value of the symplectic form ω is defined as

$$\omega(u, w) = mn' - nm'. \tag{4.3}$$

Conjugation A^* of a matrix A representing an endomorphism in $\mathbf{E}$ is the adjugate matrix, namely

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{4.4}$$

Conjugation of vectors in $\mathbf{E}$ is a map into the dual space, expressed in terms of matrices as

$$\begin{bmatrix} m \\ n \end{bmatrix}^* = \begin{bmatrix} -n & m \end{bmatrix}. \tag{4.5}$$

Now, the symplectic product may be performed via matrix multiplication: $\omega(u, w) = u^* w$. The map defined by (4.5) is the symplectic conjugation of the spinor. Also, note that $AA^* = A^*A = \det(A)I$.

Remark 4.3. In the complex representation, the symplectic product is $\omega(u, w) = \frac{i}{2}(\bar{u}w - \bar{w}u)$.

Proposition 4.4. *For any two matrices A and B and vector u in the spinor space we have:*

$$\begin{aligned} (i) \qquad & (AB)^* = B^* A^* \\ (ii) \qquad & (A)^{**} = A \\ (iii) \qquad & (Au)^* = u^* A^* \,. \end{aligned} \tag{4.6}$$

Proposition 4.5. *The group that preserves the symplectic structure (up to a sign) is the modular group $SL^{\pm}(2,\mathbb{Z}) \subset SL^{\pm}(2,\mathbb{R})$ understood here as the group of 2×2 integer (respectively, real) matrices with determinant equal ± 1.*

Proof. Preservation of the symplectic form is equivalent to matrix property $A^* A = \pm I$. Indeed:

$$\omega(Au, Aw) = (Au)^*(Aw) = u(A^*A)w = \det(A) u^* w = \pm\omega(u,w)$$

for any u, w, since $A^* A = \pm \det(A) I$, and by assumption $\det(A) = \pm 1$. Let us relate the two spaces. Given any endomorphism M of the space of triangles $\mathbb{R}^{2,1}$, we shall call an endomorphism $\widetilde{M}$ its spinor representation, if

$$M(\varphi(u)) = \varphi(\widetilde{M} u)\,. \tag{4.7}$$

Now we shall try to understand the geometry of the spin (Euclid's) representation of the Pythagorean triples in terms of the kwaternions defined in the previous sections. Recall the matrices representing the algebra:

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

By analogy to the spinor description of Minkowski space explored in theoretical physics, we shall build a "magic correspondence" for Pythagorean triples and their spinor description. First, we shall map the vectors of the space of triangles, $\mathbb{R}^{2,1}$ into the traceless 2×2 real matrices, M_{22}^0. The map will be denoted by tilde $\sim\colon \mathbb{R}^{2,1} \to M_{22}^0$ and defined:

$$\begin{aligned} \mathbf{v} = (x,y,z) \longrightarrow \widetilde{\mathbf{v}} &= x \cdot \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + y \cdot \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + z \cdot \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -y & x+z \\ x-z & y \end{bmatrix}. \end{aligned} \tag{4.8}$$

□

Proposition 4.6. *The matrix representation of the Minkowski space of triangles $\mathbb{R}^{2,1}$ realizes the original Minkowski norm via the determinant:*

$$||\mathbf{v}|| = 4 \det \widetilde{\mathbf{v}} = -x^2 - y^2 + z^2\,.$$

The scalar product may be realized by traces, namely:

$$\mathbf{v} \cdot \mathbf{w} = -2 \,\mathrm{Tr}\; \widetilde{\mathbf{v}}\, \widetilde{\mathbf{w}}$$

and particular vector coefficients may be read from the matrix by:

$$v^i = -\det \sigma_i \cdot \mathrm{Tr}\; \widetilde{\mathbf{v}} \sigma_i$$

The technique of Clifford algebras allows one to represent the orthogonal group by the corresponding pin group. Since an orthonormal transformation may be composed from orthogonal reflections in hyperplanes, one finds a realization of the action of the Lorenz group on the Minkowski space of triangles via conjugation by matrices of the spin group $SL^{\pm}(2,\mathbb{R})$; in particular, for any orthogonal matrix $A \in O(2,1;\mathbb{Z})$, the action $\mathbf{v}' = A\mathbf{v}$ corresponds to

$$\widetilde{\mathbf{v}}\,' = \widetilde{A}\,\widetilde{\mathbf{v}}\,\widetilde{A}^* \tag{4.9}$$

that is, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{v} & \xrightarrow{\ \sim\ } & \widetilde{\mathbf{v}} = \frac{1}{2}\sum v^i\sigma_i \\ \Big\downarrow A & & \Big\downarrow \text{conj } \widetilde{A} \\ A\mathbf{v} & \xrightarrow{\ \sim\ } & \widetilde{A}\widetilde{\mathbf{v}}\widetilde{A}^* \end{array}$$

And now the reward: since the Pythagorean triples lie on the "light cone" of the Minkowski space, we may construct them from spinors in a manner analogous to the standard geometry of spinors for relativity theory. But here we reconstruct Euclid's parameterization of the triples. Recall that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} m^2-n^2 \\ 2mn \\ m^2+n^2 \end{bmatrix}.$$

Thus we have:

Theorem 4.7. *The spin representation of Pythagorean triples splits into a tensor (Kronecker) product:*

$$\widetilde{v} = \frac{1}{2}\begin{bmatrix} -y & x+z \\ x-z & y \end{bmatrix} = \begin{bmatrix} -mn & m^2 \\ -n^2 & mn \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} \otimes \begin{bmatrix} -n & m \end{bmatrix}. \tag{4.10}$$

Hence we obtain yet another aspect of the Euclid's formula, namely a tensor version of the (1.2) and (4.1):

$$\widetilde{\varphi(u)} = u \otimes u^*\,. \tag{4.11}$$

Note that $[-n, m]$ in (4.10) is the symplectic conjugation of the spinor $[m, n]^T$. Now, due to the above Theorem, the adjoint action splits as follows

$$\widetilde{\mathbf{v}}\,' = \widetilde{A}\,\widetilde{\mathbf{v}}\,\widetilde{A}^* = \widetilde{A}\begin{bmatrix} m \\ n \end{bmatrix} \otimes [-n\ m]\,\widetilde{A}^* = \left(\widetilde{A}\begin{bmatrix} m \\ n \end{bmatrix}\right) \otimes \left(\widetilde{A}\begin{bmatrix} m \\ n \end{bmatrix}\right)^*. \tag{4.12}$$

And the conclusion to the story: The spin representation emerges as "half" of the above representation:

$$\begin{bmatrix} m \\ n \end{bmatrix} \to \begin{bmatrix} m' \\ n' \end{bmatrix} = \widetilde{A}\begin{bmatrix} m \\ n \end{bmatrix}. \tag{4.13}$$

Magic Correspondence		
	Minkowski space $\mathbb{R}^{2,1}$	Traceless 2×2 matrices
Main object	$\mathbf{v} = (x, y, z)$	$\widetilde{v} = \sum \mathbf{v}^i \sigma_i = \frac{1}{2}\left[\begin{smallmatrix} -y & x+z \\ x-z & y \end{smallmatrix}\right]$
Norm	$\Vert\mathbf{v}\Vert = -x^2 - y^2 + z^2$	$\Vert\mathbf{v}\Vert = 4 \det \widetilde{\mathbf{v}}$
Action	$\mathbf{v}' = A\mathbf{v}$ $(A \in O(2,1;\mathbb{Z}))$	$\widetilde{\mathbf{v}}' = \widetilde{A}\widetilde{\mathbf{v}}\widetilde{A}^*$ $(\widetilde{A} \in SL^{\pm}(2,\mathbb{Z}))$
Minkowski scalar product	$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T G \mathbf{w}$	$\mathbf{v} \cdot \mathbf{w} = -2 \,\mathrm{Tr}\, \widetilde{\mathbf{v}}\widetilde{\mathbf{w}}$
The ith coefficient	$v^i = \mathbf{v} \cdot \mathbf{e}_i$	$v^i = -\det \sigma_i \cdot \mathrm{Tr}\, \widetilde{\mathbf{v}}\sigma_1$

TABLE 1. Magic correspondence for Pythagorean triples and their spinor description

Remark 4.8 (on d-duality)**.** The "duality" (2.4) is also represented in spin language, namely, the matrix D of exchange of x with y and the corresponding 2-by-2 spin matrix, $\widetilde{D}$, are:

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \widetilde{D} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \tag{4.14}$$

Indeed, we have:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m+n \\ m-n \end{bmatrix} .$$

The d-duality may be expressed in the form of a commuting diagram

$$\begin{array}{ccc} \begin{bmatrix} m \\ n \end{bmatrix} & \xrightarrow{\ \sim\ } & \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} m^2 - n^2 \\ 2mn \\ m^2 + n^2 \end{bmatrix} \\ \Big\downarrow \widetilde{D} & & \Big\downarrow D \\ \begin{bmatrix} m+n \\ m-n \end{bmatrix} & \xrightarrow{\ \sim\ } & 2 \begin{bmatrix} y \\ x \\ z \end{bmatrix} = \begin{bmatrix} 4mn \\ 2m^2 - 2n^2 \\ 2m^2 + 2n^2 \end{bmatrix} \end{array}$$

5. Hall Matrices and Their Spinor Representation

It is known that all primitive Pythagorean triples can be generated by the following three *Hall matrices* [4]

$$U = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \quad L = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \quad R = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix} \tag{5.1}$$

by acting on the initial vector $\mathbf{v} = [3, 4, 5]^T$ ("Egyptian vector"). For clarification, a few examples:

$$Lv = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix} \quad R\mathbf{v} = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 17 \end{bmatrix}$$

$$U\mathbf{v} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 21 \\ 20 \\ 29 \end{bmatrix} \qquad \mathrm{URL}^2 U\mathbf{v} = \begin{bmatrix} 3115 \\ 3348 \\ 4573 \end{bmatrix}.$$

Let us state it formally:

Theorem 5.1 (Hall)**.** *The set of primitive Pythagorean triples is in one-to-one correspondence with the algebra of words over alphabet* $\{U, L, R\}$*.*

The original argument of Hall was algebraic (see [4] for a proof). But here we shall reinterpret this intriguing result in terms of geometry. Hall matrices may be understood in the context of our previous section and augmented with the spinor description. Let us start with this:

Proposition 5.2. *The Hall matrices and their products are elements of the Lorentz group* $O(2, 1; \mathbb{Z})$*. In particular, for any* $\mathbf{v} = [x, y, z]^T$*,*

$$g(\mathbf{v}, \mathbf{v}) = 0 \Rightarrow g(M\mathbf{v}, M\mathbf{v}) = 0,$$

and therefore they permute Pythagorean triples.

Proof. Elementary. Recall that the matrix of the pseudo-Euclidean metric is G = diag(1,1,-1). One readily checks that Hall matrices preserve the quadratic form, i.e., that $X^T GX = G$ for $X = U$, L, R. In particular we have $L, R \in SO(2, 1)$, as det $L = 1$, det $R = 1$. Since det $U = -1$, U contains a reflection. □

Hall's theorem thus says that the set of primitive Pythagorean triples coincides with the orbit through $[3, 4, 5]^T$ of the action of the semigroup generated by the Hall matrices, a subset of the Lorenz group of the Minkowski space of triangles:

$$\mathrm{gen}\{R, L, U\} \subset O(2, 1; \mathbb{Z})\,.$$

Thus the results of the previous section apply in particular to the Hall semigroup. In particular:

Theorem 5.3. *The spin representation of the Hall matrices are*

$$\widetilde{U} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \widetilde{L} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \quad \widetilde{R} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} . \tag{5.2}$$

In particular, they are members of the modular group $SL^{\pm}(2,\mathbb{Z}) \subset SL^{\pm}(2,\mathbb{R})$, that is they preserve the symplectic form up to a sign.

Proof. The transformations in the space of Euclid's parameters that correspond to the Hall matrices may be easily found with elementary algebra. By Proposition 4.5, we need to check $M^*M = \pm I$. Simple calculations show:

$$\widetilde{U}^*\widetilde{U} = -I \qquad \widetilde{L}^*\widetilde{L} = I \qquad \widetilde{R}^*\widetilde{R} = I\,.$$

That is, the determinants are $\det\widetilde{U} = -1$, $\det\widetilde{L} = 1$, and $\det\widetilde{R} = 1$. □

Clearly, the "magic correspondence" outlined in the previous section holds as well for the Hall matrices. In particular, the spin version of matrices act directly on Pythagorean spinors (4.9), and the action on the Pythagorean vectors may be obtained by the tensor product (4.13).

A classification of the semigroups in the modular group $SL^{\pm}(2,\mathbb{Z})$ that generate the set of primitive Pythagorean spinors as their orbits seems an interesting question.

Remark on the structure of the Pythagorean semigroup and its spin version. The last unresolved question concerns the origin or the geometric meaning of the Hall matrices and their spin version. First, one may try to interpret Hall matrices in terms of the Minkowski space-time structure. They may easily be split into a boost, spatial rotation and reflection. Indeed, define in $\mathbb{R}^{2,1}$ these three operators:

$$H = \begin{bmatrix} 3 & 0 & \sqrt{8} \\ 0 & 1 & 0 \\ \sqrt{8} & 0 & 3 \end{bmatrix}, \quad T = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{where } c = s = \sqrt{2}/2$$

($H =$ boost by "velocity" $(\sqrt{3/8}, 0)$, $T =$ rotation by 45°). Then the Hall matrices are the following Lorentz transformations :

$$\begin{aligned} U &= T^2(T^{-1}HT) = THT \\ R &= THTR_1 \\ L &= THTR_2 \end{aligned}$$

where $R_1 = \mathrm{diag}(-1,1,1)$ and $R_2 = \mathrm{diag}(1,-1,1)$ represent reflections. For instance, U represents a boost in the special direction (1,1) followed by space point-inversion. This path seems to lead to nowhere. Thus we may try the spinor version of the Hall matrices. One may easily see that the latter can be expressed as linear combinations of our pseudo-Pauli basis:

$$\widetilde{L} = \sigma_0 - \sigma_2 - \sigma_3 \qquad \widetilde{R} = \sigma_0 + \sigma_1 + \sigma_3 \qquad \widetilde{U} = \sigma_0 + \sigma_1 - \sigma_2\,.$$

But this naïve association does not seem to explain anything. Quite surprisingly, insight may be found in the geometry of disk packing. This is the subject of the next section.

6. Pythagorean Triangles, Apollonian Gasket and Poincaré Disk

Now we shift our attention to — at first sight rather exotic for our problem — Apollonian gaskets.

Apollonian window. Apollonian gasket is the result of the following construction. Start with a unit circle, called in the following the *boundary circle*. Inscribe two circles so that all three are mutually tangent. Then inscribe a new circle in every enclosed triangular-shaped region (see Figure 3a). Continue *ad infinitum*. A special case, when the first two inscribed circles are half the radius of the boundary circle will be called the *Apollonian window* (shown in Figure 3b). The Apollonian

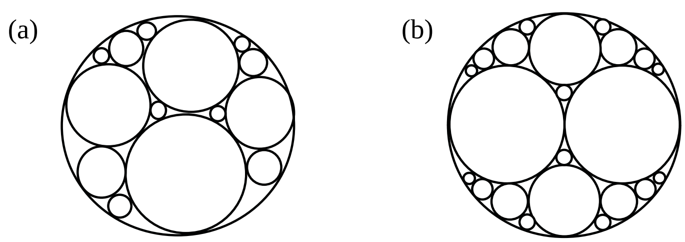

FIGURE 3. An Appollonian gasket and the Apollonian window

window has amazing geometric properties [11], [9]. One of them is the fact that the curvature of each circle is an integer. Also, as demonstrated in [8], the center of each circle has rational coordinates. It can be shown that the segment that joins the centers of any two adjacent circles forms the hypotenuse of a Pythagorean triangle, whose two legs are parallel/perpendicular to the main axes. More specifically, the sides of the triangle become integers when divided by the product r_1r_2 of the radii of the adjacent circles (see Figure 4).

Let us call a *subboundary circle* any circle in the Apollonian window tangent to the boundary circle (shaded circles in Fig. 5). Consider a pair of tangent circles of which one is the boundary— and the other a subboundary circle. If we prolong the hypotenuses of the associated triangles, we shall hit points on the boundary circle, namely the points of tangency. Due to this construction, the slope of each such line is rational. We will try to see how to permute these points.

Among the many symmetries of the Apollonian window are inversions in the circles that go through the tangency points of any three mutually tangent circles. Such inversions permute the disks of the window, and in particular preserve their

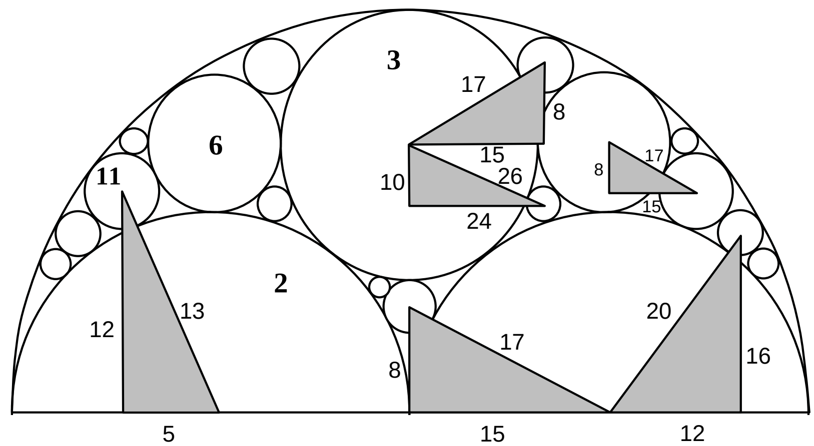

FIGURE 4. Pythagorean triangles in the Apollonian window. Bold numbers represent the curvatures of the corresponding circles.

tangencies. We shall look at the following three symmetries labeled A, B, C (see Figure 5b):

$$\begin{aligned}&A - \text{reflection through the vertical axis}\\&B - \text{reflection through the horizontal axis}\\&C - \text{inversion through the circle } C \text{ (only a quarter of the circle is shown)}\end{aligned} \tag{6.1}$$

Proposition 6.1. *The three compositions of maps*

$$CA, \quad CB, \quad CBA \tag{6.2}$$

leave the set of subboundary circles of the first quadrant invariant. In particular, they permute points of tangency on the boundary circle in the first quadrant.

Proof. A reflection in line A or B or their composition AB (reflection through the central point) carries any circle in the first quarter to one of the other three quarters. If you follow it with the inversion through C, the circle will return to the first quarter. Since tangency is preserved in these transformations, the proposition holds. □

The crucial observation is that both the lines A, B and circle C may be understood as "lines" in the Poincaré geometry, if the circle is viewed as the Poincaré disk. This will allow us to represent these operations by matrices using the well-known hyperbolic representation of the Poincaré disk.

Poincaré disk. Recall the geometry of the hyperbolic Poincaré disk. In the standard model, the set of points is that of a unit disk D in the Euclidean plane:

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \text{ with } \partial D = \{(x, y) : x^2 + y^2 = 1\}.$$

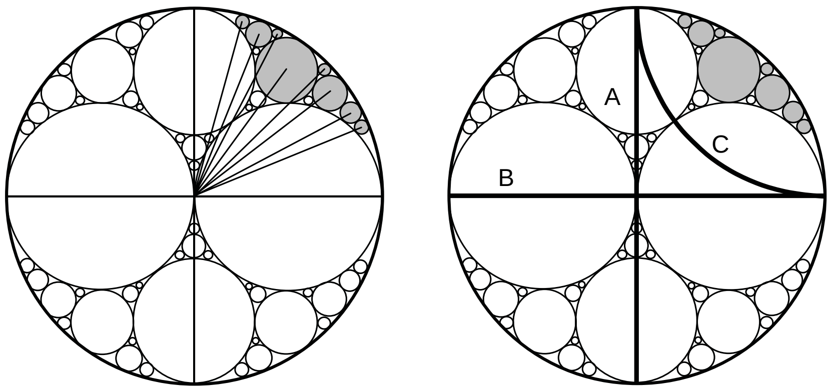

Figure 5. (a) Subboundary circles and the corresponding Pythagorean rays. (b) Three symmetries of the Apollonian window.

Poincaré *lines* are the circles that are orthogonal to ∂D (clearly, only the intersection with D counts). This geometry – as is well known – may be induced from a hyperbolic linear space. Consider a three-dimensional Minkowski space $\mathbb{R}^{2,1}$ and a hyperboloid H:

$$t^2 - x^2 - y^2 = 1\,. \tag{6.3}$$

Stereographic projection π onto the plane $P \in \mathbb{R}^{2,1}$ defined by $t = 0$, with the vertex of projection at $(-1, 0, 0)$, brings all points of the hyperboloid H onto D in a one-to-one manner. In particular, each Poincaré line in D is an image of the intersection of a plane in $\mathbb{R}^{2,1}$ through the origin $\mathbb{O}$ with the hyperboloid H, projected by π onto D. We shall use this plane-line correspondence. Recall also that reflection in a plane P can be done with the use of a unit normal vector $\mathbf{n}$:

$$R_n : \mathbf{v} \quad \rightarrow \quad \mathbf{v}' = \mathbf{v} - 2\frac{\langle \mathbf{v}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathbf{n} \tag{6.4}$$

where the orthogonality $\mathbf{n} \perp P$ and the scalar product are in the sense of the pseudo-Euclidean structure of the Minkowski space $\mathbb{R}^{2,1}$.

Back to the Apollonian window. Consider the three symmetries of the Apollonian window (6.2). Each of them may be realized in terms of a reflection in a corresponding plane in the hyperbolic representation.

Proposition 6.2. *The three symmetries* (6.1) *of the Poincaré disc (coinciding with the Apollonian window) have the following matrix representations*

$$\begin{aligned}
&\textit{Symmetry } A: \qquad \mathbf{n}_1 = [1,0,0]^T \quad \to \quad R_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\textit{Symmetry } B: \qquad \mathbf{n}_2 = [0,1,0]^T \quad \to \quad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\textit{Symmetry } C: \qquad \mathbf{n}_3 = [1,1,1]^T \quad \to \quad R_3 = \begin{bmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{bmatrix}.
\end{aligned} \tag{6.5}$$

Proof. One can easily verify that each $\mathbf{n}_i$ is unit and corresponds to the assigned symmetry. Here are direct calculations for finding the matrix corresponding to the third symmetry C corresponding to $\mathbf{n} = \mathbf{n}_3$. Acting on the basis vectors and using (6.4) we get:

$$R_3\mathbf{e}_1 = \mathbf{e}_1 - 2\frac{\langle \mathbf{e_1}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\cdot\frac{-1}{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}$$

$$R_3\mathbf{e}_2 = \mathbf{e}_2 - 2\frac{\langle \mathbf{e_2}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\cdot\frac{-1}{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -2 \end{bmatrix}$$

$$R_3\mathbf{e}_3 = \mathbf{e}_3 - 2\frac{\langle \mathbf{e_1}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2\cdot\frac{1}{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

which indeed defines matrix R_3. The other two, R_1 and R_2 are self-explanatory. □

And now we have our surprising result:

Theorem 6.3. *The hyperbolic representation of the permutations* (6.2) *of the subboundary circles of the first quadrant correspond to the Hall matrices:*

$$\begin{aligned}
R_3R_1 &= \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} = L \\
R_3R_2 &= \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix} = R \\
R_3R_1R_2 &= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} = U\,.
\end{aligned} \tag{6.6}$$

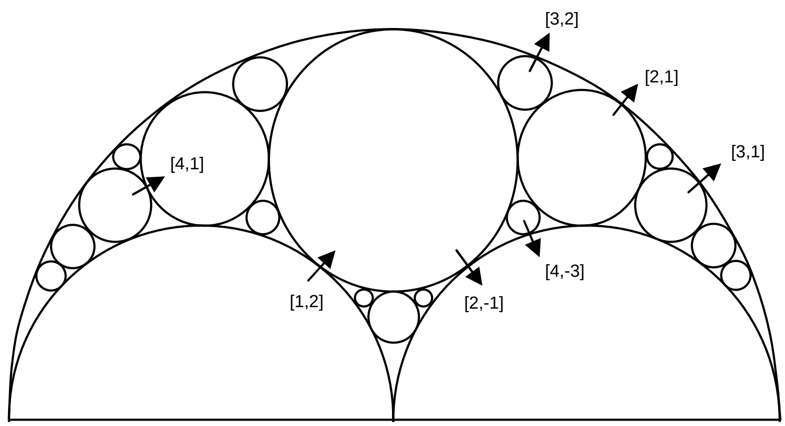

FIGURE 6. Spinors for some Pythagorean triangles in the Apollonian window, given as labels at the points of tangency of the corresponding pairs of circles.

Corollary. *Each primitive Pythagorean triangle is represented in the Apollonian window.*

This finally answers our question of Section 5 on the origin and structure of the matrices of spin representation of the Hall matrices. Following the above definitions, we get in Clifford algebra the following representations of the reflections that constitute the symmetries (6.2):

$$\begin{aligned}
(\mathbf{f}_1+\mathbf{f}_2+\mathbf{f}_3)\mathbf{f}_1 &= \mathbf{f}_1^2+\mathbf{f}_2\mathbf{f}_1+\mathbf{f}_3\mathbf{f}_1 = -\sigma_3+\sigma_0-\sigma_2 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \widetilde{L} \\
(\mathbf{f}_1+\mathbf{f}_2+\mathbf{f}_3)\mathbf{f}_2 &= \mathbf{f}_1\mathbf{f}_2+\mathbf{f}_2^2+\mathbf{f}_3\mathbf{f}_2 = \sigma_0+\sigma_3+\sigma_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \widetilde{R} \\
(\mathbf{f}_1+\mathbf{f}_2+\mathbf{f}_3)(-\mathbf{f}_3) &= -\mathbf{f}_1\mathbf{f}_3-\mathbf{f}_2\mathbf{f}_3-\mathbf{f}_3^2 = -\sigma_2+\sigma_1+\sigma_0 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \widetilde{U}
\end{aligned} \tag{6.7}$$

(cf. (5.2)). Note that the duality map (4.13) that exchanges x and y in the Pythagorean triangles and which has a spinor representation $D = \sigma_1 - \sigma_2$ fits the picture, too, as the vector $\mathbf{n} = [1, -1, 0]$ represents (scaled) reflection in the plane containing the x-y diagonal.

We conclude our excursion into the spin structure of the Euclid parameterization of Pythagorean triples with Figure 6 that shows spinors corresponding to some Pythagorean triangles in the Apollonian window.

7. Conclusions and Remarks

We have seen that the space of Euclid's parameters for Pythagorean triples is endowed with a natural symplectic structure and should be viewed as a spinor space for the Clifford algebra $\mathbb{R}_{2,1}$, built over 3-dimensional Minkowski space, whose integer light-like vectors represent the Pythagorean triples (see Figure 7). The minimal algebra for $\mathbb{R}^{2,1}$ is four-dimensional and may be conceptualized as "pseudo-quaternions" $\mathbb{K}$ and represented by 2×2 matrices. In this context the Pythagorean triples may be represented as traceless matrices, and Euclid's parameterization map as a tensor product of spinors. This set-up allows us to build the spinor version of the Hall matrices. The Hall matrices acquire a geometric interpretation in a rather exotic context of the geometry of the Apollonian window.

Euclid's discovery of the parameterization of Pythagorean triples may be viewed then as the first recorded use of a spinor space. The spinor structure of the Apollonian window is another interesting subject that will be studied further elsewhere.

The method may be generalized to other dimensions, as indicated here:

Method A. Define a Pythagorean (k,l)-tuple as a system of $(k+l)$ integers that satisfy

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_k^2 = b_1^2 + b_2^2 + b_3^2 + \ldots + b_l^2 \tag{7.1}$$

In order to obtain a parameterization of Pythagorean (k,l)-tuples do the following: Start with pseudo-Euclidean space $\mathbb{R}^{k,l}$, build a representation of the Clifford algebra $\mathbb{R}_{k,l}$. Then split the matrix that represents the isotropic vectors of $\mathbb{R}^{k,l}$ into a tensor product of spinors. This tensor square provides the parameterization when restricted to the integer spinors.

Example. Consider Pythagorean quadruples, that is quadruples of integers (a,b,c,d)

$$a^2 + b^2 + c^2 = d^2\,.$$

Among the examples are (1,2,2,3), (1,4,8,9), (6,6,7,11), etc. A well-known formula that produces Pythagorean quadruples is [10]:

$$\begin{aligned} a &= 2mp \\ b &= 2np \\ c &= p^2 - (m^2+n^2) \\ d &= p^2 + m^2 + n^2\,. \end{aligned} \tag{7.2}$$

It is also known that not all quadruples are generated this way, for instance (3, 36, 8, 37) is excluded [18].

Let us try our method. The spinor representation of the Clifford algebra $\mathbb{R}_{3,1}$ is well known. The light-like vectors are represented by the Hermitian 2×2 matrices that split into a spinor product,

$$M = \begin{bmatrix} d+a & b+ci \\ b-ci & d-a \end{bmatrix} = 2 \begin{bmatrix} z \\ w \end{bmatrix} \otimes \begin{bmatrix} \bar{z} & \bar{w} \end{bmatrix}, \tag{7.3}$$

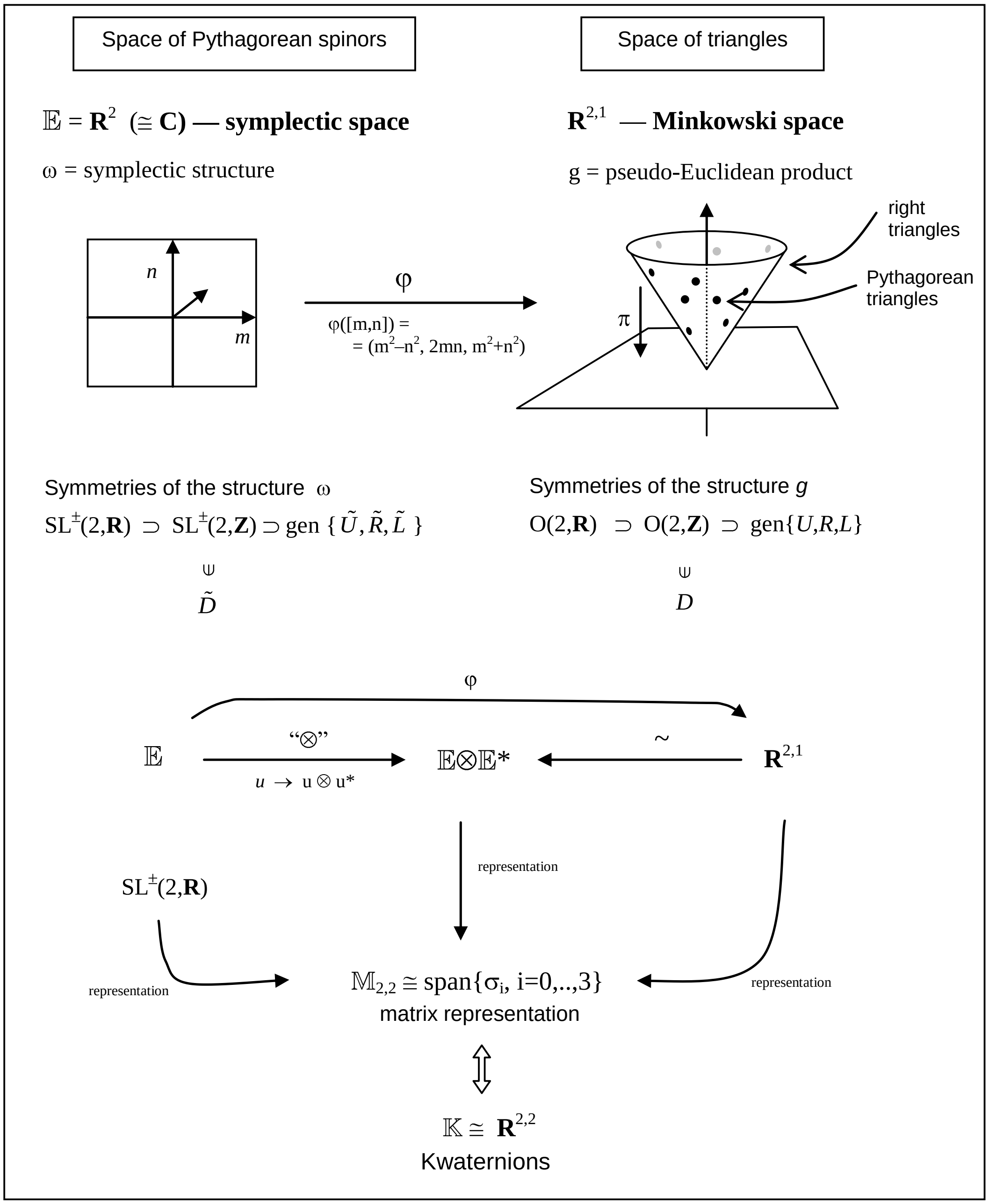

FIGURE 7. Objects related to Euclid's parametrization of Pythagorean triples.

where the matrix M on the left side is well-known in physics with d standing for time and a, b, c for spatial variables. Note that indeed $\det M = d^2 - a^2 - b^2 - c^2$ represents the quadratic form of Minkowski space. The factor of 2 is chosen to keep things integer.

Let $z = m + ni$ and $w = p + qi$. Then (7.3) becomes:

$$\begin{bmatrix} d+a & b+ci \\ b-ci & d-a \end{bmatrix} = 2 \begin{bmatrix} m^2+n^2 & (mp+nq)+(np-mq)i \\ (mp+nq)-(np-mq)i & p^2+q^2 \end{bmatrix}$$

which may be readily resolved for a, b, c, d. Thus we have proven:

Theorem 7.1. *The following formulae produce all Pythagorean quadruples:*

$$\begin{aligned} a &= m^2+n^2-p^2-q^2 \\ b &= 2(mp+nq) \\ c &= 2(np-mq) \\ d &= m^2+n^2+p^2+q^2 \end{aligned} \tag{7.4}$$

that is

$$(m^2+n^2-p^2-q^2)^2 + (2mp+2nq)^2 + (2nn-2mq)^2 = (m^2+n^2+p^2+q^2)^2\,.$$

The quadruple (3, 36, 8, 37) that was not covered by (7.2) may be now obtained by choosing $(m, n, p, q) = (4, 2, 4, 1)$

$$(4^2+2^2-4^2-1^2)^2 + (2\cdot4\cdot4+2\cdot2\cdot1)^2 + (2\cdot2\cdot4-2\cdot4\cdot1)^2 = (4^2+2^2+4^2+1^2)^2\,.$$

The standard Euclid's parameterization of Pythagorean triples results by choosing $n = q = 0$ (which imposes $c = 0$), or by $m = n$ and $p = q$, although this time with a redundant doubling in the formulae. Also, choosing only $q = 0$ will result in the system of formulae (7.2).

Quite similarly, we can treat Pythagorean hexads using the fact that $\mathbb{R}_{5,1} \cong \mathbb{H}(2)$ (quaternionic 2×2 matrices).

Theorem 7.2. *Let us use collective notation* $m = (m_0, m_1, m_2, m_3)$ *and* $n = (n_0, n_1, n_2, n_3)$. *Also denote in ususal way* $mn = m_0n_0 + m_1n_1 + m_2n_2 + m_3n_3$. *The following formulae produce Pythagorean hexads:*

$$\begin{aligned} a_0 &= m^2+n^2 \\ a_1 &= 2(n_0m_1 - n_1m_0 + m_3n_2 - m_2n_3) \\ a_2 &= 2(n_0m_2 - n_2m_0 + m_1n_3 - m_3n_1) \\ a_3 &= 2(n_0m_3 - n_3m_0 + m_2n_1 - m_1n_2) \\ a_4 &= 2mn \\ a_5 &= m^2 - n^2 \end{aligned} \tag{7.5}$$

that is for any $m, n \in \mathbb{Z}^4$ *we have*

$$a_0^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2\,.$$

Proof. A general Hermitian quaternionic matrix can be split into the Kronecker product

$$\begin{bmatrix} a_0 + a_5 & a \\ \bar{a} & a_0 - a_5 \end{bmatrix} = 2 \begin{bmatrix} p \\ q \end{bmatrix} \otimes \begin{bmatrix} \bar{p} & \bar{q} \end{bmatrix}, \tag{7.6}$$

where on the left side $a_0, a_5 \in \mathbb{R}$, and $a = a_1 + a_2 i + a_2 j + a_3 k \in \mathbb{H}$, and on the right side $q = m_0 + m_1 i + m_2 j + m_3 k$ and $p = n_0 + n_1 i + n_2 j + n_3 k$. Resolving this equation gives (7.5). □

For example $(m, n) = ((1, 2, 2, 1), (2, 1, 1, 1))$ produces $2^2 + 3^2 + 4^2 + 8^2 + 14^2 = 17^2$. Note that in this parameterization map, $\varphi : \mathbb{Z}^8 \to \mathbb{Z}^6$, the dimension of the parameter space outgrows that of the space of Pythagorean tuples.

Reconsidering the entries of matrices of equation (7.6) gives us this alternative generalization:

Theorem 7.3. *Consider equation*

$$\begin{bmatrix} a + b & c \\ \bar{c} & a - b \end{bmatrix} = 2 \begin{bmatrix} p \\ q \end{bmatrix} \otimes \begin{bmatrix} \bar{p} & \bar{q} \end{bmatrix}, \tag{7.7}$$

with $a, b \in \mathbb{R}$, and $c, p, q \in \mathbb{A}$, where $\mathbb{A}$ is an algebra with a not necessarily positive definite norm and cojugation satisfying $aa^* = |a|^2$ and $(ab)^* = b^* a^*$. If the quadratic form of $\mathbb{A}$ is of signature (r, s), then formula (7.5) produces Pythagorean $(r + 1, s + 1)$-tuples.

Note that as a special case we may use the Cliford algebras themselves as algebra $\mathbb{A}$. As an example consider "duplex numbers" $\mathbb{D}$, [7], which form Clifford algebra of $\mathbb{R}^1$. Let $c = c_0 + c_1 I \in \mathbb{D}$, where $c_0, c_1 \in \mathbb{R}$, and $I^2 = 1$ (pseudo-imaginary unit). Similarly, set $p = p_0 + p_1 I$ and $q = q_0 + q_1 I$. Then (7.7) produces Pythagorean (2,2)-tuples:

$$(p_0^2 - p_1^2 + q_0^2 - q_1^2)^2 + (2p_0 q_1 - 2p_1 q_0)^2 = (p_0^2 - p_1^2 - q_0^2 + q_1^2)^2 + (2p_0 q_0 - 2p_1 q_1)^2$$

A simple application that goes beyond Clifford algebras: using the algebra of octonions, $\mathbb{A} = \mathbb{O}$, results in a parameterization $\varphi : \mathbb{Z}^{16} \to \mathbb{Z}^{10}$ of Pythagorean "decuples" by 16 parameters.

Remark 7.4. Since Clifford algebras are 2^n-dimensional, using them in Theorem 7.3 will lead to parameteriztion of (generalized) Pythagorean $(2^n + 2)$-tuples by 2^{n+1} "Euclid's parameters". For $n = 0, 1, 2, \ldots$ we get 3-, 4-, 6-, 10-, ..., -tuples. Incidentally, these numbers occur frequently in various string theories, the reason for which is not fully understood [19].

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