# Golden window

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The design of the arch window illustrated in FIGURES 1 and 2 should please every fan of geometry. With this window in my home, whether the circular medallion of FIGURE 1 or semi-circular arch of FIGURE 2, I would offer guests a puzzle: start with two small central circles of unit diameter. Then find the radius $R$ of the two circles on their left and right, given that a a pair of congruent circles (dotted) that are simultaneously tangent to all the other circles.

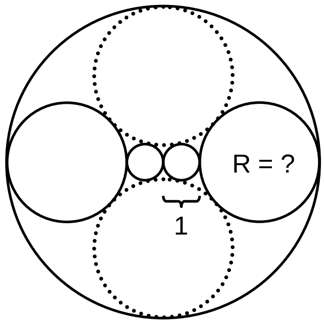



**Figure 1** A puzzle.

Guests could deduce, by multiple application of the Pythagorean Theorem, for instance, that $R = \varphi \approx 1.618$, the golden ratio!

There is more: the centers of the two circles of radius $R$ are located at distance $1+\varphi = \varphi^2$ from the center of the window and the radius of the big circumscribing circle is the cube of the golden ratio, $1+2\varphi = \varphi^3$. Actually, the figure is replete with the golden ratio and its powers; hence the design deserves the name **golden window**.

I could spend time calling my guests' attention to its "golden" attributes of the window. To start with, the window contains powers of the golden ratio from $\varphi^0$ to $\varphi^4$, as shown in FIGURE 2. It also contains various segments with **golden cuts**, as shown in the same figure below the window.

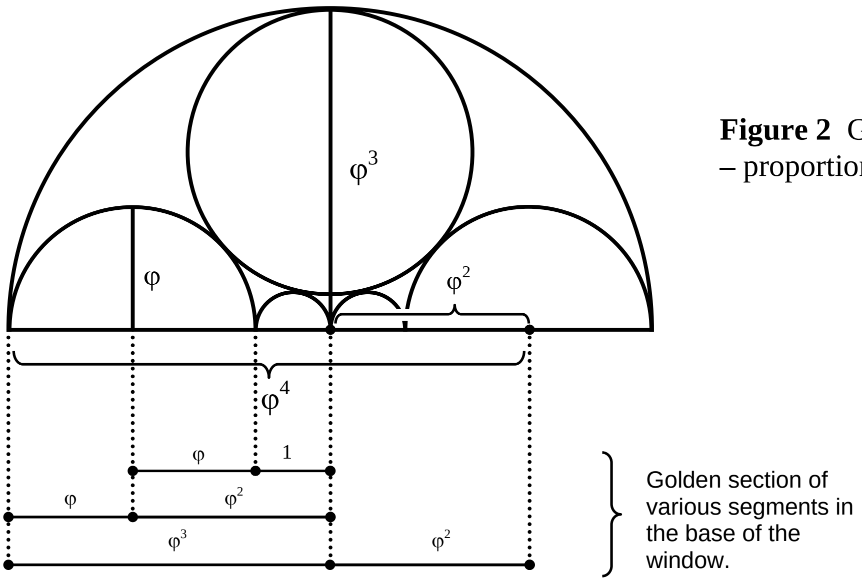



**Figure 2** Golden Window – proportions.

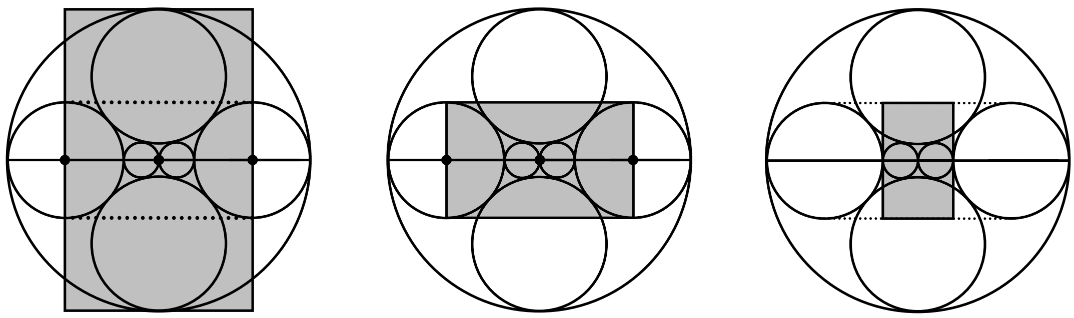
Recognizing such segments is an easy game (once you establish that $R = \varphi$) if only you remember the fundamental properties of the golden ratio, namely

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2} \qquad \text{and} \qquad \varphi^n = \mathrm{F}_n\varphi + \mathrm{F}_{n-1}\ ,$$

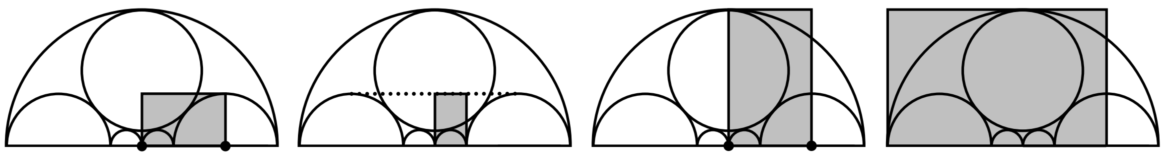
where $F_n$ denotes the $n$-th Fibonacci number, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, *etc*., with $F_{n+1} = F_n + F_{n-1}$. For small $n$ we have:

$$\begin{aligned}
\varphi^2 &= \varphi^1 + \varphi^0 = \varphi + 1\\
\varphi^3 &= \varphi^2 + \varphi = 2\varphi + 1\\
\varphi^4 &= \varphi^3 + \varphi^2 = 3\varphi + 2\\
\varphi^5 &= \varphi^4 + \varphi^3 = 5\varphi + 3, \qquad \textit{etc.}
\end{aligned}$$

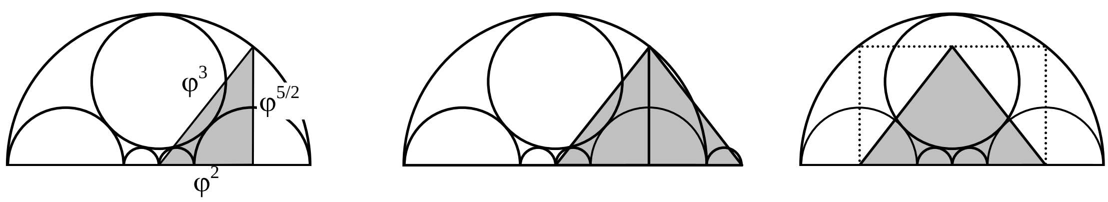
Next, I would point to various **golden rectangles** in the construction:



If the window were truncated to the upper half, I would expect my guests to spot these golden rectangles:



The culmination would be the challenge of finding the silhouette of the **Khu-fu pyramid of Giza**. Recall that the pyramid's half-silhouette makes (intentionally or not) a nearly perfect model of the so-called Kepler's triangle, a right triangle whose edges form a geometric progression. The only such triangle has sides proportional to $1 : \sqrt{\varphi} : \varphi$. The shaded triangle shown below at the left has just such proportions.



Indeed, its height $h$ can be calculated from its base $\varphi^2$ and its hypotenuse $\varphi^3$ with the Pythagorean theorem:

$$h^2 = (\varphi^3)^2 - (\varphi^2)^2 = \varphi^6 - \varphi^4 = \varphi^4(\varphi^2 - 1) = \varphi^4\varphi = \varphi^5 .$$

Thus we have the triangle $(\varphi^2, \varphi^{5/2}, \varphi^3) = \varphi^2(1, \sqrt{\varphi}, \varphi)$ — Kepler's golden triangle scaled by the factor $\varphi^2$. The pyramid may of course be drawn in a central position as well (the trick to see it is to apply reflective symmetry to the initial triangle).

A last challenge would be to consider the two small circles in the upper left and right of the window. The question is: are their centers collinear with the center of the other upper circle? And are they vertically aligned with the circles below them, or do they only seem so? The emerging rectangle (dotted lines) seems to be composed of two squares (the center of either small upper circle and the principal center would form a square's diagonal); is it indeed a square?

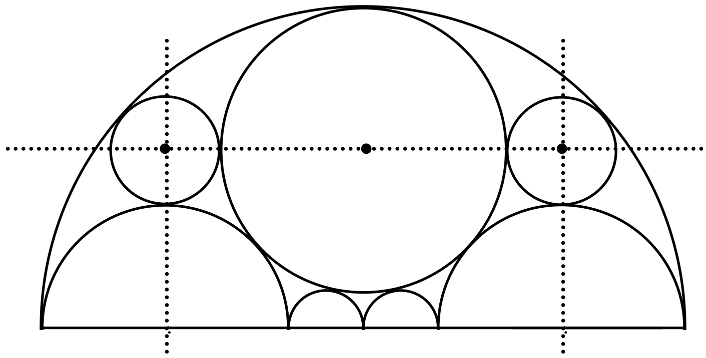


**Figure 3** Are the centers aligned?

This would lead us to consider Descartes' circle formula [**1**], its extension [**7**], and its generalization [**5**]. But that would have to wait until after dinner.

## Tools for tangent circles

Readers may have solved the original puzzle—to find the radii of the circles that make the construction possible—by repeated use of the Pythagorean Theorem, but the last few questions present quite a computational challenge if this is the only tool available.. A more insightful approach to circles in various configurations starts with Descartes theorem, its extension (which was discovered only in 2001), and finally the most general theorem. They are collected below for the convenience of the reader and as an inducement to study further the beautiful geometry of circles.

**Level 1: Descartes theorem**

In 1643, René Descartes gave a remarkable formula that relates the radii of four mutually tangent circles [**2**]:

$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2 = 2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right). \qquad (1)$$

Using the *reciprocals* of radii, *i.e.*, *curvatures*, the formula reads

$$(a+b+c+d)^2 = 2\,(a^2+b^2+c^2+d^2)\,, \qquad (2)$$

where $a=1/r_1$, $b=1/r_2$, *etc*. It is assumed that if a circle *contains* the other circles, its curvature is negative.

Descartes' formula has been rediscovered many times and its higher-dimensional generalization has also been found [**1**, **10**, **4**]. A system of four pairwise tangent circles is called the **Descartes configuration**, and sometimes **Soddy's circles** [**8**], after one of the re-discoverers [**10**].

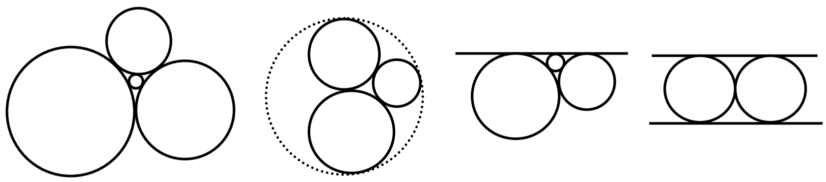


**Figure 4** Examples of four circles in the Descartes configuration

One could use Descartes' formula to determine the radius of the upper corner circles in FIGURE 3. They each belong to a Descartes configuration together with three other circles of curvatures

$$a = \varphi^{-1}, \quad b = \sqrt{5}\varphi^{-3}, \quad c = -\varphi^{-3}.$$

Substituting in (2) we get $d = \sqrt{5}\,\varphi^{-1}$, which gives the radius $r = \varphi/\sqrt{5} = (5+\sqrt{5})/10$. This suffices to establish the co-linearity of points hypothesized in the puzzle, except that one would need first to know the radius of the central upper circle.

**Level 2: Extended Descartes theorem**

Note that Descartes' formula is quadratic and may be represented in matrix form. If $b_1 = 1/r_1$, $b_2 = 1/r_2$, *etc*., denote curvatures then

$$\begin{bmatrix} b_1 & b_2 & b_3 & b_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \ , \qquad (3)$$

or — briefly — $B^T DB = 0$, with the obvious association of symbols. The **Extended Descartes theorem** was proposed in 2002 in [**7**]. In addition to the curvatures, it includes the positions of the centers $(x_i, y_i)$, $i = 1,\ldots,4$, and some additional variables, yet to be explained. First let us enjoy the nice matrix form:

$$\begin{bmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_3 \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 & \dot{y}_4 \\ b_1 & b_2 & b_3 & b_4 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 & \bar{b}_4 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \dot{x}_1 & \dot{y}_1 & b_1 & \bar{b}_1 \\ \dot{x}_2 & \dot{y}_2 & b_2 & \bar{b}_2 \\ \dot{x}_3 & \dot{y}_3 & b_3 & \bar{b}_3 \\ \dot{x}_4 & \dot{y}_4 & b_4 & \bar{b}_4 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 8 & 0 \end{bmatrix} . \qquad (4)$$

Note that the original Descartes formula (3) is embedded in (4). The dotted variables represent "reduced coordinates" — reduced by the corresponding radii: $\dot{x}_i = x_i/r_i$ and $\dot{y}_i = y_i/r_i$ . The barred $b$s denote the "co-curvatures" of the circles and are defined as $\bar{b} = (\dot{x}^2 + \dot{y}^2 - 1)/b$ for each circle, but they need not concern us: For our purposes one needs only to extract from (4) three equations, $X^T DX = -4$, $Y^T DY = -4$, and $B^T DB = 0$, where $X$, $Y$, and $B$ denote the first three columns of the third matrix, respectively.

**Level 3: General circle theorem**

Unfortunately the crucial circles in the Golden Window do not form a Descartes configuration. The question is: is there a formula that would apply to not-necessarily-tangent circles? I am happy to report that there is.

Suppose you have four circles in general position (some tangent, some possibly orthogonal, *etc.*). Define a "circle configuration matrix" $f$ with entries

$$f_{ij} = \frac{d_{ij}^2 - r_i^2 - r_j^2}{2 r_i r_j} \ . \qquad (5)$$

The six numbers $d_{ij}$ denote the distances between the centers of the corresponding circles.

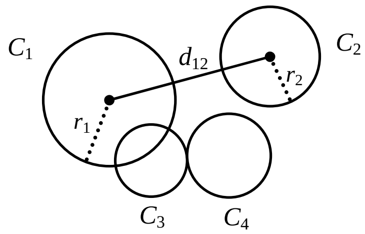



**Figure 5** Four circles in general position.

**Theorem (Circle Configuration Theorem) [6]**: With the above notation, four circles in general position satisfy

$$\begin{bmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_3 \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 & \dot{y}_4 \\ b_1 & b_2 & b_3 & b_4 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 & \bar{b}_4 \end{bmatrix} \begin{bmatrix} F_{11} & & \cdots & F_{14} \\ & & & \\ \vdots & & & \vdots \\ F_{41} & & \cdots & F_{44} \end{bmatrix} \begin{bmatrix} \dot{x}_1 & \dot{y}_1 & b_1 & \bar{b}_1 \\ \dot{x}_2 & \dot{y}_2 & b_2 & \bar{b}_2 \\ \dot{x}_3 & \dot{y}_3 & b_3 & \bar{b}_3 \\ \dot{x}_4 & \dot{y}_4 & b_4 & \bar{b}_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}. \tag{6}$$

or $AFA^T = G$, where $F$ is the inverse of the configuration matrix, $F = f^{-1}$ .

The truncated version for curvatures only is thus $B^T F\, B = 0$, or

$$\sum_{i,j} F_{ij}\, b_i\, b_j = 0 \tag{7}$$

and may be viewed as a strong generalization of the Descartes formula.

Fortunately, finding the entries of the matrix $f$ is often quite simple and direct, without the need of equation (4). Special cases are shown in FIGURE 6, where the ***ij***-th entry is denoted as a "product of two circles", $f_{ij} = \langle C_i, C_j \rangle$, called in [**5**] the "Pedoe product", since it may indeed be traced to D. Pedoe [**9**, p. 155].

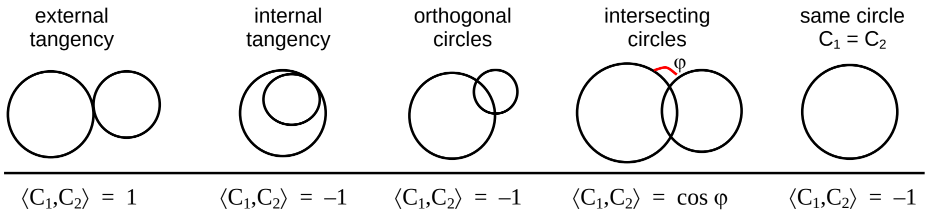



**Figure 6** Pedoe inner product of two circles (possible entries of matrix *f*)

Note that in the special case of mutually tangent circles, FIGURE 6, matrix $f$ is the one in equation (3). Its inverse is $F = f^{-1} = 4f$ ; thus the Descartes formula (including the extended version) follows as a very special case.

The theorem may be used to solve the puzzle. *Nota bene*, the design is a special case of a "lens chain" – a collinear system of tangent circles simultaneously tangent to two congruent disks; more on this may be found in [**6**].

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**Summary** Finding appearances of the golden ratio in various nooks and crannies of mathematics brings delight, often surprise. This note presents, in the form of a puzzle, a configuration of circles that is replete with the golden ratio. But that is only the surface. One tool to analyze such figures is the "master matrix equation" that rules circle (and *n*-sphere) configurations. This equation generalizes the famous circle theorem of Descartes (known also as Soddy's kissing circle theorem).

## Questions answered

The first two questions posed at the end of this note have positive answers: the centers of the little corner circles are indeed aligned with the centers of the adjacent circles. Their exact positions and radii are shown in the figure below.

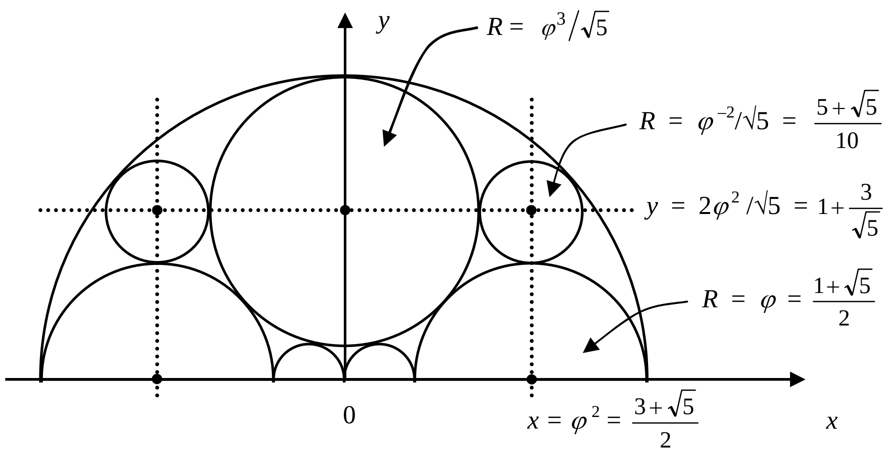


**Figure A-4** Some answers

As to the "square", it turns out that it is actually a rectangle of proportion $2 : \sqrt{5}$, as can be seen above.

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