

# Duplex Numbers, Diffusion Systems, and Generalized Quantum Mechanics

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We show that the relation between the Schrödinger equation and diffusion processes has an algebraic nature and can be revealed via the structure of “duplex numbers.” This helps to clarify that quantum mechanics cannot be reduced to diffusion theory. Also, a generalized version of quantum mechanics where  $\mathbb{C}$  is replaced by a normed algebra with a unit is proposed.

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## INTRODUCTION

The idea that extending the formalism of quantum mechanics beyond the field of complex numbers may bring about some additional interesting structure has been investigated since the 1960s (Finkelstein *et al.*, 1962; Emch, 1963; Nash and Joshi, 1992; Adler, 1995; and references therein). The argument that the extended field must be a division algebra limits the investigations via the Frobenius (1878) theorem to three fields: real numbers  $\mathbb{R}$ , the complex plane  $\mathbb{C}$ , and quaternions  $\mathbb{H}$ . One can argue, however, that the object replacing complex numbers may be any associative algebra with a quadratic norm, not necessarily positive definite. This leads to a wide spectrum of interesting possibilities, among which the Clifford algebras may serve as the simplest generalization (encompassing quaternions and duplex numbers investigated below).

Nagasawa (1993) introduced a certain type of diffusion system and showed that it can replace the Schrödinger equation under appropriate change of potential. This observation led him to conclude that quantum mechanics can be demystified and replaced by a more “intuitive” and less mysterious diffusion theory.

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In this paper we introduce “quantum mechanics” over duplex numbers, i.e., with the imaginary unit  $I$  satisfying  $I^2 = +1$ . We show that the Nagasawa type of diffusion systems may be understood within the quantum formalism over the field of duplex numbers. In this framework, we can see why, contrary to some claims, the standard quantum mechanics cannot be reduced to a diffusion system, and the argument is topological in nature. [This is relevant to the question of whether “one needs  $i$  in quantum mechanics” (Jauch, 1973; Jammer, 1974).]

The example presented here shows also that the formalism of quantum mechanics may be extended beyond the paradigm of division fields  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$ . Some comments on generalized quantum mechanics follow.

## 1. THE ALGEBRA OF DUPLEX NUMBERS

Duplex numbers were introduced by Clifford (1873, 1878) as “double numbers” and they have recently gained interest among physicists [see, e.g., Hucks (1993) for the relation to Dirac spinors and Kunstatter *et al.* (1983) for applications in the theory of gravitation].

By duplex numbers we understand the algebra  $\mathbb{D} = \{a + bI \mid a, b \in \mathbb{R}\}$  with  $I^2 = +1$ . In many respects,  $\mathbb{D}$  is similar to the field of complex numbers  $\mathbb{C}$ , except that the elements of the form  $a \pm aI$  do not have an inverse. The conjugation of a duplex number  $z = a + bI$  is defined as  $\bar{z} = a - bI$ , and a norm is defined via

$$|z|^2 = z\bar{z} = (a + bI)(a - bI) = a^2 - b^2 \quad (1.1)$$

Thus, duplex numbers  $\mathbb{D}$  form a plane  $\mathbb{R}^2 = \text{span}\{1, I\}$  with hyperbolic geometry. The inverse of  $z$  is  $z^{-1} = \bar{z}/|z|^2$ . Duplex numbers can be expressed in the polar form

$$z = \rho e^{I\varphi} = \rho(\cosh \varphi + I \sinh \varphi) \quad (1.2)$$

with  $\tanh \varphi = b/a$  and  $\rho \in \mathbb{R}$  for numbers of positive square norm,  $|z|^2 > 0$ , and  $\rho \in I \cdot \mathbb{R}$  otherwise. In particular, a hyperbolic version of de Moivre’s formula holds:

$$\rho_1 e^{I\varphi_1} \cdot \rho_2 e^{I\varphi_2} = \rho_1 \rho_2 e^{I(\varphi_1 + \varphi_2)} \quad (1.3)$$

Duplex numbers manifest two-dimensional space-time structure in an algebraic form. In particular, elements of form

$$e^{I\varphi} \quad (1.4)$$

represent hyperbolic rotations of the plane  $\mathbb{D}$  (“boosts”) and represent the connected component of the Lie group  $SO(1, 1)$  containing the group unit.

## 2. QUANTUM MECHANICS OVER DUPLEX NUMBERS

The standard Schrödinger equation of quantum mechanics over complex numbers is

$$i\partial_t\Psi + \frac{1}{2}\Delta\Psi - U \cdot \Psi = 0 \tag{2.1}$$

( $m = 1, \hbar = 1$ ), where  $\Psi: \mathbb{R}^n \rightarrow \mathbb{C}$  is a complex-valued wave function, the Laplacian  $\Delta = \nabla \cdot \nabla$  is a composition of gradients, and  $U: \mathbb{R}^n \rightarrow \mathbb{R}$  is a potential. We can express the wave function in polar (logarithmic) coordinates as  $\Psi = e^{R+iS}$ . It is a well-known result (Nelson, 1966; Pelce; 1996) that the Schrödinger equation resolves into a pair of partial differential equations for the real-valued functions  $R$  and  $S$ :

$$\begin{cases} -\partial_t S + \frac{1}{2}\Delta R + \frac{1}{2}(\nabla R)^2 - \frac{1}{2}(\nabla S)^2 - U = 0 \\ \partial_t R + \frac{1}{2}\Delta S + \nabla S \cdot \nabla R = 0 \end{cases} \tag{2.2}$$

Indeed, let  $\Psi = e^z$ . Then (2.1) is equivalent to  $i\partial_t z + \frac{1}{2}\Delta z + \frac{1}{2}(\nabla z)^2 - U = 0$ . Now, substituting  $z = R + iS$  and separating the real and the imaginary parts gives the two equations of (2.2).

Consider now an analog of the Schrödinger equation over duplex numbers,

$$I\partial_t\phi + \frac{1}{2}\Delta\phi - W \cdot \phi = 0 \tag{2.3}$$

for some potential  $W: \mathbb{R}^n \rightarrow \mathbb{R}$ . Assume  $\phi = e^{R+iS}$ . Then the duplex Schrödinger equation resolves into a pair of diffusion-type equations

$$\begin{cases} \partial_t S + \frac{1}{2}\Delta R + \frac{1}{2}(\nabla R)^2 + \frac{1}{2}(\nabla S)^2 - W = 0 \\ \partial_t R + \frac{1}{2}\Delta S + \nabla S \cdot \nabla R = 0 \end{cases} \tag{2.4}$$

This result may be restated in a notation that encompasses both the complex and the duplex case. A quantum wave function  $\Psi: \mathbb{R}^n \rightarrow \mathbb{R}^2$  assumes values in two-dimensional algebra with a unit, spanned by  $\{1, a\}$ .

*Theorem 1.* Define a (generalized) Schrödinger equation

$$\mathcal{G}[a; U] \Psi [a; R, S] = 0 \tag{2.5a}$$

where  $\mathcal{G}$  is an operator based on an (invertible) algebraic unit  $a$ ,

$$\mathcal{G}[a; U] = a^{-1}\partial_t + \frac{1}{2}a^{-2}\Delta + U \cdot \tag{2.5b}$$

and the generalized wave function has the polar form,

$$\Psi[a; R, S] = e^{R+aS} \tag{2.5c}$$

Then if  $a^2 = -1$ ,  $\mathcal{G}[a, U]$  is the standard Schrödinger operator and (2.5a)

describes a quantum system with potential  $U$ . If  $a^2 = 1$ , then  $\mathcal{G}[a, W]$  is the “duplex Schrödinger operator” describing the diffusion system (2.4).

The question is whether these two systems can be equivalent. Comparison of (2.2) and (2.4) leads immediately to the following.

*Proposition 2.* The solutions to the two equations

$$\begin{aligned} \text{(i)} \quad \mathcal{G}[i, U] \Psi[i, R, S] &= 0, & i^2 &= -1 \\ \text{(ii)} \quad \mathcal{G}[I, W] \Psi[I, R', S'] &= 0, & I^2 &= +1 \end{aligned} \tag{2.6}$$

coincide, i.e.,  $S \equiv S'$  and  $R \equiv R'$ , if

$$W = U + 2\partial_\rho S + (\nabla S)^2 \tag{2.7a}$$

or, equivalently, if

$$W = -U + \Delta R + (\nabla R)^2 \tag{2.7b}$$

The equivalence of Schrödinger’s equation (2.1) and the diffusion system (2.4) upon condition (2.7a) is studied by Nagasawa (1993). Here, we have established that this relation is algebraic and originates in the opposition of the complex versus duplex numbers.

Here is an algebraic difference between  $\mathbb{D}$ - and  $\mathbb{C}$ -quantum mechanics.

*Corollary 3.* The duplex Schrödinger equation (2.4) can be written in an isotropic form

$$\begin{cases} \partial_t(S + R) + \frac{1}{2}\Delta(S + R) + \frac{1}{2}(\nabla(S + R))^2 = W \\ \partial_t(S - R) + \frac{1}{2}\Delta(S - R) + \frac{1}{2}(\nabla(S - R))^2 = 0 \end{cases} \tag{2.8a}$$

which, by introducing vectors  $\mathbf{Z} = [S + R, S - R]$  and  $\mathbf{W} = [W, 0]$ , we can express as

$$\partial_t \mathbf{Z} + \frac{1}{2} \Delta \mathbf{Z} + \frac{1}{2} (\nabla \mathbf{Z})^2 = \mathbf{W} \tag{2.8b}$$

This can easily be verified directly from (2.4). Notice that the standard (complex) Schrödinger system does not admit such a representation. The symmetry of (2.8) is due to the geometry of duplex numbers; indeed, two isotropic elements

$$\gamma = \frac{1}{\sqrt{2}}(1 + I), \quad \bar{\gamma} = \frac{1}{\sqrt{2}}(1 - I)$$

satisfy the following multiplication table:

$$\gamma^2 = \gamma \quad \bar{\gamma}^2 = \bar{\gamma}, \quad \gamma \cdot \bar{\gamma} = 0$$

Units  $\gamma$  and  $\bar{\gamma}$  determine the isotropic coordinates (“light cone”) on  $\mathbb{D}$  (algebraically, each generates an ideal in  $\mathbb{D}$ ). For any element  $z = \alpha\gamma + \beta\bar{\gamma}$ , one has

$$\bar{z} = \beta\gamma + \alpha\bar{\gamma}$$

with the norm

$$z\bar{z} = (\alpha\gamma + \beta\bar{\gamma})(\beta\gamma + \alpha\bar{\gamma}) = \alpha\beta(\gamma + \bar{\gamma}) = \sqrt{2} \alpha\beta$$

Now, we can see that Eq. (2.8) corresponds to the split  $R + IS = (1/\sqrt{2})(R + I)\gamma + (1/\sqrt{2})(R - I)\bar{\gamma}$ .

It is easy to see that the above two cases exhaust all two-dimensional cases. Let the ‘imaginary’ element  $a$  be of a general form, such that  $a^2 = \alpha + a\beta = [\alpha, \beta]$ . Setting  $\alpha \neq 0$  assures invertibility of  $a$ . Any specification of the vector  $[\alpha, \beta]$  is equivalent to one of two cases,

$$\begin{aligned} [-1, 0] &\rightarrow \text{standard QM} \quad (a = i) \\ [+1, 0] &\rightarrow \text{diffusion process} \quad (a = I) \end{aligned} \tag{2.9}$$

discussed above (2.2), (2.4). Indeed, transformation of basis  $\{1, \mathbf{a}\}$  into a new basis  $\{1, \mathbf{f}\}$  with  $\mathbf{f}$  defined as

$$\mathbf{f} = \begin{cases} (2\mathbf{a} - \beta)/\sqrt{\beta^2 + 4\alpha} & \text{if } \alpha \neq -\beta^2/2 \\ \mathbf{a} - \beta/2 & \text{otherwise} \end{cases}$$

establishes an algebra isomorphism to one of the cases (2.9), since  $\mathbf{f}^2 = 1 \cdot \text{sgn}(\beta^2 + 4\alpha)$ .

Direct calculations show that in the general basis, (2.5) separates into a system of two equations:

$$\begin{aligned} (i) \quad &\partial_t R + \frac{1}{2}\Delta S + \nabla S \cdot \nabla R + \beta(\partial_t S + \frac{1}{2}(\nabla S)^2 + U) = 0 \\ (ii) \quad &\frac{1}{2}\Delta R + \frac{1}{2}(\nabla R)^2 + \alpha(\partial_t S + \frac{1}{2}(\nabla S)^2) = 0 \end{aligned} \tag{2.10}$$

For the sake of illustration, consider the idempotent case  $[0, 1]$ , i.e.,  $a^2 = a$ :

$$\begin{cases} \Delta R + (\nabla R)^2 = 2W \\ \partial_t(R + S) + \frac{1}{2}\Delta S + \frac{1}{2}(\nabla S)^2 + \nabla S \nabla R = 0 \end{cases} \tag{2.11}$$

The equivalence of (2.11) with the standard Schrödinger system (in the sense of Proposition 2) can be ensured by  $W \equiv U$  and  $2\partial_t S + (\nabla S)^2 = 0$ . This case is isomorphic to the diffusion case  $[1, 0]$ , as a substitution  $b = 2a - 1$  shows, since  $b^2 = 4a^2 - 4a + 1 = 1$ .

### 3. DOES QUANTUM PHYSICS NEED $\sqrt{-1}$ ?

It has been argued (Nagasawa, 1993) that the quantum formalism can be reduced to a study of diffusion processes and that the equivalence of Schrödinger’s equation to diffusion systems “demystifies” quantum mechan-

ics (see also Collins, 1992). Let us look at this proposition. Since the diffusion systems in consideration can be viewed in terms of duplex algebra, we can now restate the question: “Can quantum mechanics be rewritten in terms of duplex numbers?” That is to say, are quantum formalisms with imaginary units  $a^2 = \pm 1$  equivalent in their ability to describe concrete physical systems?

Many features of the “duplex Schrödinger mechanics” invite one to advocate such a view; let us review some of them. First, the probability density in the probabilistic interpretation can be obtained via an equivalent of the familiar Born formula

$$\Psi\Psi = e^{(R+IS)}e^{(R-IS)} = e^{2R} \quad (3.1)$$

which is analogous to the standard complex version. The additivity of phases also holds, due to (1.2). Note also that the second equations of both versions of Schrödinger’s equation, the complex (2.2) and the duplex (2.4), coincide; both represent the “continuity equation,” which can be rewritten as

$$\partial_t P + \nabla \mathbf{J} = 0 \quad (3.2)$$

where  $P$  and  $\mathbf{J}$  are the scalar-valued “density” and vector-valued “current,” respectively. In the duplex case, they are defined as real functions (in position and time):

$$P = \Psi\Psi = e^{2R} \quad \mathbf{J} = \frac{1}{2I} (\Psi\nabla\Psi - \Psi\nabla\Psi) = e^{2R}\nabla S \quad (3.3)$$

Thus, the “kinematic” components of the two Schrödinger formalisms, complex and duplex, coincide. The other (first) equations in (2.2) and (2.4) are extended versions of the Jacobi–Hamilton equation of classical mechanics, and differ in the two cases, complex and duplex. The question is whether a particular dynamical system can be expressed equally well by either of them. The free particle satisfies the equivalence condition (2.7) trivially,  $W = U = 0$ , and

$$\Psi \sim e^{I(px - Et)} \quad (3.4)$$

with  $E = p^2/2$ . Similarly, one can replace the complex (quantum) form by the duplex (diffusion) form for any stationary system. Indeed, assume that  $\Psi$  is separable

$$\Psi(t, \mathbf{x}) = e^{-IEt} \cdot u(\mathbf{x}) \quad (3.5)$$

Here we have  $\partial_t S = -E = \text{const}$  and  $\partial_t R = 0$ , hence the second (continuity) equation of (2.2) is satisfied automatically. The duplex Schrödinger equation (2.4) reduces to its time-independent version

$$\frac{1}{2}\nabla u(\mathbf{x}) = (E + W(\mathbf{x})) \cdot u(\mathbf{x}) \quad (3.6)$$

Compare it with the standard Schrödinger equation

$$\frac{1}{2}\nabla u(\mathbf{x}) = (U(\mathbf{x}) - E) \cdot u(\mathbf{x}) \quad (3.7)$$

These two differ only in interpretation of the energy; an identification  $W = U - 2E$  makes the two descriptions, complex and duplex, equivalent. For instance,  $W = x^2/2$  leads to the Hermite polynomials, as  $U = -x^2/2$  does in the standard quantum mechanics.

These simple cases may indeed suggest that the two descriptions are interchangeable. However, the equivalence breaks down once one goes beyond the configuration spaces of trivial topology, and the reason lies in (i) the superposition principle and (ii) the different topology of the unit circles in  $\mathbb{C}$  and  $\mathbb{D}$ , or the different “symmetry groups”:

$$SO(2) = \{e^{i\phi}\} \sim S^1 \subset \mathbb{C} \quad (\text{compact}) \quad (3.8)$$

$$SO\uparrow(1,1) = \{e^{t\phi}\} \sim \mathbb{R} \subset \mathbb{D} \quad (\text{noncompact})$$

The essence of the formalism of quantum mechanics lies in its ability to deal with systems, where a number of states can coexist in superposition and interfere at the time of observation. Consider the classic Young two-slit experiment as an example. Standard cursory estimations (with zero potential) of the intensity at position  $x$  on the screen near its center ( $x = 0$ ) leads to a sinusoidal pattern

$$\left|e^{i\varphi} + e^{i(\varphi+\delta)}\right|^2 \sim \cos^2(\delta/2) \quad (3.9)$$

where the phase difference from the slits is approximately

$$\delta = x \cdot d/L$$

with  $d$  the distance between slits and  $L$  the distance to the screen ( $d \ll L$ ). A similar estimation (with zero potential) for the duplex wave gives

$$\left|e^{t\varphi} + e^{t(\varphi+\delta)}\right|^2 \sim \cosh^2(\delta/2) \quad (3.10)$$

which has one maximum at the center and vanishes as  $|x|$  increases.

The fringe pattern in the two-slit experiment arises from the phase periodicity and therefore cannot be explained by diffusion equations. (In the complex case, phase  $S$  develops modulo  $2\pi$  on the circle, while in the duplex case, it may develop in an unbounded manner.) For this reason, results of any path-split type of experiment (involving configuration space with a nontrivial fundamental group) cannot be explained within the framework of duplex quantum mechanics.

In conclusion, the interpretational problems of quantum mechanics cannot be resolved in terms of diffusion processes (beyond simple cases of

topologically trivial configuration spaces). These two phenomena, quantum processes and diffusion processes, are different in nature as they correspond to different nonisomorphic algebras  $\mathbb{C}$  and  $\mathbb{D}$  with topologically different symmetry groups.

The above discussion is relevant to the old question of whether “quantum mechanics requires the imaginary unit  $i$ ” [see summaries of the discussion in Jauch (1973) and Jammer (1974)]. It was first posed by P. Ehrenfest and (unsatisfactorily) addressed by Pauli (1933). Later, Stueckelberg *et al.* (1960; Stueckelberg and Guenin, 1961) considered real Hilbert spaces and showed that the uncertainty principle requires a superselection rule that is equivalent to a complex structure in the Hilbert space. The present paper points to the topological nature of the problem and to the compactness of  $S^1 \subset \mathbb{C}$  as the source of the experimental results (“fringes”) in the topologically nontrivial configuration spaces.

#### 4. QUANTUM MECHANICS GENERALIZED

A possible generalization of the formalism of quantum mechanics has been advocated for some time, but has typically been limited to the division algebras  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$  (Adler, 1995). Starting from the classical dynamical interpretation of the Schrödinger equation (Strochi, 1966; Rowe *et al.*, 1980; Heslot, 1985; Jones, 1992), Millard (1997) has recently proposed a generalization to the case of an associative ring with a conjugation. The present paper also encourages one to go beyond this paradigm and consider a quantum formalism based on algebras with a norm relaxed from the condition of positive definiteness. Here we present a general outline; a more detailed exposition will be developed elsewhere.

Let  $A$  be an associative algebra with a unit and a (quadratic) norm  $\|\cdot\|^2$  that is not necessarily positive definite (and possibly degenerate). We shall demand that

$$\|ab\|^2 = \|a\|^2 \cdot \|b\|^2 \quad (4.1)$$

which is dictated by the correspondence principle (Adler, 1995). In an  $A$ -quantum mechanics, the Hilbert space is replaced by a right  $A$ -module with a set of  $A$ -valued operators acting on the left. In particular, the canonical commutation rules

$$[\hat{p}, \hat{x}] = a \cdot \hbar \cdot \text{id} \quad (4.2)$$

for some fixed invertible  $a \in A$  (hereafter  $\hbar = m = 1$ ) are consistent with the Schrödinger representation



$$\hat{x} = x \cdot, \quad \hat{p} = a^{-1}\hbar\nabla, \quad \hat{H} = -a^{-1}\hbar\partial_t \tag{4.3}$$

For instance, a particle in potential  $U$  is described by

$$-a^{-1} \partial_t \Psi = \frac{1}{2}a^{-2}\Delta\Psi + U\Psi \tag{4.4}$$

Clifford algebras (Porteous, 1981) present an example of normed algebras with a unit and therefore determine a family of generalized forms of quantum mechanics. Let  $V = \mathbb{R}^{(p,q)}$  be an  $n$ -dimensional space equipped with a pseudo-Euclidean structure of signature  $(p, q)$  where  $n = p + q$ . The corresponding Clifford algebra  $\mathbb{R}_{(p,q)}$  can be viewed as a  $2^n$ -dimensional Grassmann space  $A = \wedge V$  with the algebra product induced from

$$vw = -g(v, w) + v \wedge w \tag{4.5}$$

for  $v, w \in V \subset A$  [in particular,  $vv = -g(v, v)$ ]. Let  $\{e_i\}$  be an orthonormal basis in  $V$ . Denote the ordered set of its indices by  $I = \{1, 2, \dots, n\}$ . Any index subset  $A \subset I$  defines a basis element  $e_A$  of  $A$ ; for instance,

$$e_\phi = 1, \quad e_{\{i\}} = e_i \quad e_{\{1,3,7\}} = e_1e_3e_7 \quad e_I = e_1e_2 \dots e_n$$

where  $\phi$  denotes the empty subset.

Now, let us consider Clifford quantum mechanics. It seems natural (but not necessary) to chose  $a$  of Eq. (4.3) to be the volume element (pseudoscalar)  $a = e_I$  (clearly,  $e_I^2 = \pm 1$ ). Thus the commutation rules (4.2) read  $[\hat{p}, \hat{x}] = e_I \text{id}$ , and the Schrödinger representation  $\hat{x} = x \cdot, \hat{p} = e_I^{-1}\nabla$  gives, after multiplying both sides by  $e_I^2$ ,

$$-e_I\partial_t\Psi = \frac{1}{2}\Delta\Psi \pm U\Psi \tag{4.6}$$

where the undetermined sign is that of  $e_I^2 = (-1)^{n(n-1)/2+p}$ . Due to the noncommutativity of Clifford algebras, the polar form of the Schrödinger equation does not emerge naturally. The Schrödinger equation resolves into  $2^n$  intertwined equations of a general form

$$\partial_t\Psi_A = \pm\frac{1}{2}\Delta\Psi_{A^c} \pm U\Psi_{A^c}$$

labeled by subsets  $A \subset I$ , where  $A^c = I - A$ . Intuitively, subspaces with  $e_A^2 = 1$  correspond to “diffusion sectors,” and those with  $e_A^2 = -1$  correspond to “quantum sectors.”

The two algebras  $\mathbb{C}$  and  $\mathbb{D}$  juxtaposed in this paper correspond to two cases of Clifford algebras based on one-dimensional spaces (with  $e_1$  identified with  $i$  or  $I$ , respectively). The low-dimensional cases encompass the following:

- $\mathbb{R}_{1,0} \rightarrow$  standard quantum mechanics
- $\mathbb{R}_{0,1} \rightarrow$  diffusion system

$\mathbb{R}_{0,0} \rightarrow$  heat equation ( $a = 1$ )

$\mathbb{R}_{2,0} \rightarrow$  quaternionic quantum mechanics

In particular, quaternions correspond to the Clifford algebra of  $\mathbb{R}^{(2,0)}$  and have a basis  $i = e_1, j = e_2, k = e_1 \wedge e_2$ . One can express the wave function as a sum of two “complex”-valued functions

$$\Psi = \psi + k\phi = (\psi_1 + i\psi_2) + k(\phi_1 + i\phi_2) \quad (4.7)$$

with  $\psi_1, \psi_2, \phi_1,$  and  $\phi_2$  real. Then, assuming  $U$  real or complex, the generalized Schrödinger equation resolves into a pair

$$\begin{aligned} \partial_t \psi &= -\frac{1}{2} \Delta \phi + U \phi \\ -\partial_t \phi &= -\frac{1}{2} \Delta \psi + U \psi \end{aligned} \quad (4.8)$$

A detailed description of Clifford quantum mechanics involves commutation rules of Clifford algebras, which goes beyond the scope of this paper and will be presented elsewhere.

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