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Representations of the Schrödinger algebra and Appell systems

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We investigate the structure of the Schrödinger algebra. Two constructions are given that yield the physical realization via general methods starting from the abstract Lie algebra. Representations are found on a Fock space with basis given by a canonical Appell system. Generalized coherent states are used in the construction of the Hilbert space of functions on which certain commuting elements act as self-adjoint operators. This yields a probabilistic interpretation of these operators as random variables. An interesting feature is how the semidirect product structure of the Lie algebra is reflected in the probability density function. A Leibniz function and orthogonal basis for the Hilbert space are found. Then certain evolution equations connected with canonical Appell systems on this algebra are shown.

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1 Introduction

The Schrödinger Lie algebra plays an important role in mathematical physics and its applications. It has been introduced and investigated as the algebra of symmetries of the free Schrödinger equation. (see, e.g., [3, 4, 12, 14]). The (1+1) case was noticed to represent a low-dimensional Wick algebra, and as such to be isomorphic to the 2-photon algebra [2, 6]. The resulting structure of the semidirect product of the Heisenberg algebra and sl(2) was investigated in [9].

Unitary irreducible (projective) representations of the Schrödinger group were studied in [15]. A classification of the irreducible lowest weight representations of Schrödinger Lie algebra was found via the technique of singular vectors in [7]. The main feature of the present paper is to present representations of the Schrödinger algebra in an alternative way using generalized Appell systems (see Sects. 2, 3).

We begin in Sect. 2 with some preliminaries on Wick products and Appell polynomials. In Sect. 3 we give some interpretations of the notion of 'Appell systems'. Sect. 4 contains basic details of our approach to representations of the Schrödinger algebra. In particular, we show how it is built and determine a standard form. Some group calculations are done using a matrix realization of the algebra. It is remarked how to recover the physical realization of the algebra from a general approach. In Sect. 5 we construct canonical Appell systems and find a family of probability distributions associated to the Schrödinger algebra that reflects its Lie algebraic structure. In particular, we see that the results of [7] on polynomial representations based on lowest weight modules fit into our picture. The details of the associated Hilbert space comprise Sect. 6. This starts with computing the Leibniz function. We show how to recover the Lie algebra from the Leibniz function and obtain an orthogonal basis for the Hilbert space. Here again the physical realization is found in a natural way. In the final section, we show how to construct Appell systems which provide solutions to generalized heat equations on the Schrödinger algebra, corresponding to classical two-dimensional real Lévy processes.

2 Wick products and Appell polynomials

Appell polynomials are closely related to Wick products. The presentation below closely follows [1].

Let $X_1, X_2, \ldots$ be a sequence of random variables. The Wick powers are defined inductively as follows. For $k = 0$, the Wick power is equal to 1. Denoting expectation values by $\langle\quad\rangle$, for $k > 0$, $:X_1, X_2, \ldots, X_k:$ is defined recursively by

$$\langle : X_1, X_2, \ldots, X_k : \rangle = 0$$

and

$$\frac{\partial}{\partial X_i} : X_1, X_2, \ldots, X_k : \quad = \; : X_1, \ldots, X_{i-1}, \hat{X}_i, X_{i+1}, \ldots, X_k :$$

where $\hat{X}_i$ denotes the absence of the variable X_i.

Example 2.1 The first two Wick products are

$$\begin{aligned} : X_1 : &= X_1 - \langle X_1 \rangle \\ : X_1, X_2 : &= X_1 X_2 - X_1 \langle X_2 \rangle - X_2 \langle X_1 \rangle + 2 \langle X_1 \rangle \langle X_2 \rangle - \langle X_1 X_2 \rangle . \end{aligned}$$

The Appell polynomials $P_n(x)$ are then defined by

$$P_n(x) = P_{X,n}(x) = \underbrace{: X, \ldots, X :}_{n \text{ times}} \Big|_{X=x} .$$

Example 2.2 Take X with mean $\mu_1 = \langle X \rangle = 0$ and $\mu_i = \langle X^i \rangle$, $i = 2, 3, \ldots$. Then

$$\begin{aligned}
P_0(x) &= 1\,,\\
P_1(x) &= x\,,\\
P_2(x) &= x^2 - \mu_2\,,\\
P_3(x) &= x^3 - 3\mu_2 x - \mu_3\,,\\
P_4(x) &= x^4 - 6\mu_2 x^2 - 4\mu_3 x + 6\mu_2^2 - \mu_4\,,\\
P_5(x) &= x^5 - 10\mu_2 x^3 - 10\mu_3 x^2 + 5x(6\mu_2^2 - \mu_4) + 20\mu_2\mu_3 - \mu_5\,.
\end{aligned}$$

Remark 2.3 If $X \sim N(0,1)$, then we get the Hermite polynomials. However, Appell polynomials are not necessarily orthogonal polynomials. Appell polynomials $P_n(x)$, $n \in \mathbf{N}$, are characterized by the two conditions

1. $P_n(x)$ is a polynomial of degree n,
2. $\frac{\mathrm{d}}{\mathrm{d}x} P_n(x) = n\, P_{n-1}(x)\,.$

Interesting examples are furnished by *moment polynomials*,

$$P_n(x) = \int_{-\infty}^{\infty} (x+y)^n \,\mathrm{p}(dy)$$

where p is a probability measure on $\mathbf{R}$ with all moments finite. In the Gaussian case these are "heat polynomials," closely related to the Hermite polynomials. In [10] the probabilistic interpretation of Appell polynomials is used to define their analog on Lie groups where, in general, they are no longer polynomials. This explains the terminology *Appell systems*.

3 Appell systems: some interpretations

Here are three interpretations of the notion of 'Appell systems':

1. Appell systems in the classical sense. We will look at these below in connection with symmetry algebras.
2. Canonical Appell systems associated to a Lie algebra. The modifier "canonical" refers to the fact that the Appell system forms a basis for a representation of the Heisenberg-Weyl algebra, i.e., it is generated by boson creation and annihilation operators. One uses the Lie algebra to construct a Hilbert space with the Appell system as basis. This Hilbert space is a type of Fock space, with finitely many degrees of freedom. See Sect. 4.
3. General Appell systems on Lie groups. Here one uses the Lie algebra and group structure as a 'black box' into which a classical stochastic process goes in and produces a 'Lie response' – typically a process consisting of iterated stochastic integrals of the input process ([10, 11]). We will not pursue this direction in the present paper.

Now consider the first point of view.

Let D denote the differentiation operator. We may think of the space of polynomials of degree not exceeding n as the space of solutions, $\mathcal{Z}_n$, to the equation $D^{n+1}\psi = 0$. In this context an *Appell system* is defined to be a sequence of nonzero polynomials $\{\psi_0, \psi_1, \ldots, \psi_n, \ldots\}$ satisfying:

1. $\psi_n \in \mathcal{Z}_n, \forall n \geq 0$,
2. $D\psi_n = \psi_{n-1}$, for $n \geq 1$.

(Note that this differs slightly from the usual definition, as given above, Remark 2.3, which has $D\psi_n = n\psi_{n-1}$). By analogy, for any operator V, the *canonical lowering operator,* we define a V-Appell system as follows. Set

$$\mathcal{Z}_n = \{\, \psi : \; V^{n+1}\psi = 0 \,\}$$

for $n \ge 0$. Then the *V-Appell space decomposition* is the system of embeddings $\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \ldots$, and a *$V$-Appell system* is a sequence of nonzero functions $\{\psi_0, \psi_1, \ldots, \psi_n, \ldots\}$ satisfying:

1. $\psi_n \in \mathcal{Z}_n, \forall n \ge 0\,,$
2. $V\psi_n = \psi_{n-1}$, for $n \ge 1\,.$

Typically, one starts with a 'standard Appell system', such as $\psi_n = x^n/n!$, for $V = D$. Then Appell systems are generated from the standard one via time-evolution. To accomplish this for V-Appell systems, the symmetry algebra of V comes into play.

If V is an operator acting on a space of smooth functions, its *unrestricted symmetry algebra* is the Lie algebra $\mathfrak{g}(V)$ of vector fields, X, such that there exists an operator $\Lambda(X)$ in the center of $\mathfrak{g}(V)$ with

$$[X, V] = \Lambda(X)\, V\,.$$

If we require Λ to be multiplication by a scalar function, we shall talk of the *restricted* symmetry algebra, as in [7]. If we consider only those X for which $\Lambda(X) = 0$, we have the *strict* symmetry algebra $\mathfrak{g}_0(V) \subset \mathfrak{g}(V)$. Clearly, $V \in \mathfrak{g}_0(V)$. Also, it is clear that $\mathfrak{g}_0(V)$ contains the center of $\mathfrak{g}(V)$.

Proposition 3.1 *The strict symmetry algebra contains the derived algebra of unrestricted symmetries:* $\mathfrak{g}'(V) \subset \mathfrak{g}_0(V)$. *That is,* $Y \in \mathfrak{g}'(V)$ *implies* $[Y, V] = 0$.

Proof. Let $[X_1, V] = \Lambda(X_1)\, V$ and $[X_2, V] = \Lambda(X_2)\, V$. Then, by the Jacobi identity,

$$\begin{aligned}
[[X_1, X_2], V] &= [X_1, \Lambda(X_2)\, V] + [\Lambda(X_1)\, V, X_2] \\
&= \Lambda(X_2)\; [X_1, V] + \Lambda(X_1)\, [V, X_2] \\
&= \Lambda(X_2)\Lambda(X_1)\, V - \Lambda(X_1)\Lambda(X_2)\, V = 0\,,
\end{aligned}$$

using the property that the Λ operators are central. □

The relevance for V-Appell systems is this.

Proposition 3.2 *The unrestricted symmetry algebra* $\mathfrak{g}(V)$ *of an operator* V *preserves the Appell space decomposition* $\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \ldots$, *that is,* $X\mathcal{Z}_n \subset \mathcal{Z}_n$ *for every* $X \in \mathfrak{g}(V)$.

Proof. Write $[X, V] = \Lambda(X)\, V$ in the form $VX = (X - \Lambda(X))V$. Fix $n \ge 0$ and let $\psi \in \mathcal{Z}_n$. Since $\Lambda(X)$ commutes with V, we have $V^{n+1}X\psi = (X - (n+1)\Lambda(X))V^{n+1}\psi = 0$. □

New Appell systems are generated from a given one by the adjoint action of a group element generated by a 'Hamiltonian' – a function of elements of the symmetry algebra. The structure of the spaces $\mathcal{Z}_n$ is preserved, while the Appell systems provide 'polynomial solutions' to the evolution equation corresponding to the Hamiltonian. Indeed, if H is a function of operators in $\mathfrak{g}(V)$, with $H\psi_0 = 0$, then $h_n = \exp(tH)\psi_n$ will be an Appell system. For each n, the function h_n satisfies $u_t = Hu$, with $u(0) = \psi_n$. In the simplest situation where H is a function of D and the initial Appell sequence is $\psi_n = x^n/n!$, different choices of H yield many of the classically important sequences of polynomials (with perhaps minor variations).

In [7], a hierarchy of solutions to $\mathbf{S}^{(p/2)}\psi = 0$ is developed for the Schrödinger operator $\mathbf{S}$. The representations discussed there can be viewed as $\mathbf{S}$-Appell systems in the above sense. These correspond to finite-dimensional representations of sl(2) in the standard form of the Schrödinger algebra given below.

4 Schrödinger algebra

Referring to [7] for details, we recall the ($n = 1$, centrally-extended) Schrödinger algebra $\mathcal{S}_1$. Here we give the *physical realization* in terms of operators, vector fields and multiplication by functions of x and t, with their corresponding physical meaning/transformations indicated. Note that m and d are given parameters. The identity operator is denoted by I.

$$\begin{array}{lll} M = m\,\mathrm{I} & \text{mass}\,, & \\ K = t^2\partial_t + t\,x\partial_x + \frac{m}{2}x^2 - td & \text{special conformal transformation}\,, & \\ G = t\partial_x + mx & \text{Galilei boost}\,, & \\ D = 2t\partial_t + x\partial_x - d & \text{dilation (not differentiation!)}\,, & \\ P_x = \partial_x & \text{spatial translation}\,, & \\ P_t = \partial_t & \text{time translation} & \end{array} \tag{1}$$

which satisfy commutation relations given by the following multiplication table

$$\begin{array}{c|cccccc} & M & K & G & D & P_x & P_t \\ \hline M & 0 & 0 & 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 & -2\,K & -G & -D \\ G & 0 & 0 & 0 & -G & -M & -P_x \\ D & 0 & 2\,K & G & 0 & -P_x & -2\,P_t \\ P_x & 0 & G & M & P_x & 0 & 0 \\ P_t & 0 & D & P_x & 2\,P_t & 0 & 0 \end{array}\,.$$

Note that the elements $\{M, G, P_x\}$ span a Heisenberg-Weyl subalgebra, while $\{K, D, P_t\}$ span an sl(2) subalgebra. This fact, that the Schrödinger algebra is a semidirect product

$$\mathcal{S}_1 \cong \mathcal{H} \oplus_s sl(2)$$

is the basis for analyzing the representations of the Schrödinger algebra. We continue with the $n = 1$ case and indicate briefly how the case $n > 1$ goes at the end of the discussion of the standard form, since the rotation generators, J_{ij}, do not appear in the case $n = 1$.

4.1 Structural decomposition for Fock calculus

In general, in order to construct representations, we first seek a *generalized Cartan decomposition* of the Schrödinger algebra into a triple of subalgebras $\mathfrak{g} = \mathcal{P}\oplus\mathcal{K}\oplus\mathcal{L}$ where $\mathcal{P}$ and $\mathcal{L}$ are abelian, and $\mathcal{K}$ normalizes both $\mathcal{P}$ and $\mathcal{L}$:

$$[\mathcal{K},\ \mathcal{L}] \subset \mathcal{L}\,, \qquad [\mathcal{K},\ \mathcal{P}] \subset \mathcal{P}\,, \qquad [\mathcal{P},\ \mathcal{L}] \subset \mathcal{K}\,.$$

The main idea is that elements of $\mathcal{P}$ and $\mathcal{L}$ act as raising and lowering operators, respectively (cf. [13, p. 31]). The possibility of finding a scalar product in which each element of $\mathcal{P}$ has a corresponding adjoint in $\mathcal{L}$ is important, since we wish to construct a family of selfadjoint operators that provide commuting quantum observables – classical random variables in the probabilistic interpretation. In many cases, this family arises by conjugating elements of $\mathcal{P}$ by a group element with a generator from $\mathcal{L}$. This technique may be viewed as an extension of the Cayley transform for symmetric spaces. Notice that for this to work, the subalgebras $\mathcal{P}$ and $\mathcal{L}$ must be in one-to-one correspondence, the *Cartan involution* in the theory of symmetric spaces.

The Schrödinger algebra $\mathfrak{g} = \mathcal{S}_1$ admits the following generalized Cartan decomposition:

$$\{M, K, G\} \oplus \{D, P_x\} \oplus \{P_t\}\,. \tag{2}$$

Note however that $\mathcal{P}$ and $\mathcal{L}$ cannot be put into 1–1 correspondence and therefore this is of no direct use for us.

We use instead the following decomposition:

$$\underbrace{\{K, G\}}_{\mathcal{P}} \oplus \underbrace{\{M, D\}}_{\mathcal{K}} \oplus \underbrace{\{P_t, P_x\}}_{\mathcal{L}} \,. \tag{3}$$

Even though the decomposition (3) is not technically a Cartan decomposition (the requirement $[\mathcal{P},\ \mathcal{L}] \subset \mathcal{K}$ is not satisfied), it will lead to interesting results for representations of the Schrödinger algebra.

We take $R_1 = K$ and $R_2 = G$ as raising operators, and M as (multiplication by) a scalar m. The operator P_x, not properly in "Cartan's $\mathcal{L}$" of eq. (2), will be used as the lowering operator dual to G. Thus we have lowering operators $L_1 = P_t$ and $L_2 = P_x$.

4.2 A matrix representation and group calculations

A 4-dimensional representation (see [6]) of the Schrödinger algebra ($n = 1$) is given by embedding into su(4). Let X denote a typical element of the Lie algebra. Set,

$$X = a_1 m + a_2 K + a_3 G + a_4 D + a_5 P_x + a_6 P_t = \begin{pmatrix} 0 & a_5 & a_3 & 2a_1 \\ 0 & a_4 & a_2 & a_3 \\ 0 & -a_6 & -a_4 & -a_5 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4}$$

Using this basis, a typical group element may be parametrized via *coordinates of the second kind* $\{A_i\}$ as

$$g(A_1, A_2, A_3, A_4, A_5, A_6) = \\ \exp(A_1 m) \exp(A_2 K) \exp(A_3 G) \exp(A_4 D) \exp(A_5 P_x) \exp(A_6 P_t) \,.$$

In particular, the group element corresponding to (4) is

$$g(A_1, A_2, A_3, A_4, A_5, A_6) = \mathrm{e}^{-A_4} \cdot \begin{pmatrix} \mathrm{e}^{A_4} & A_5 \mathrm{e}^{A_4} - A_3 A_6 & A_3 & 2A_1 \mathrm{e}^{A_4} - A_3 A_5 \\ 0 & \mathrm{e}^{2A_4} - A_2 A_6 & A_2 & -A_3 \mathrm{e}^{A_4} - A_2 A_5 \\ 0 & -A_6 & 1 & -A_5 \\ 0 & 0 & 0 & \mathrm{e}^{A_4} \end{pmatrix} .$$

Proposition 4.1 *The coordinates of the second kind, $(A_1, \ldots, A_6)$, of a group element g given as a matrix (g_{ij}) are:*

$$A_1 = -\tfrac{1}{2} \frac{\begin{vmatrix} g_{13} & g_{14} \\ g_{33} & g_{34} \end{vmatrix}}{g_{33}} , \qquad A_2 = \frac{g_{23}}{g_{33}} , \qquad A_3 = \frac{g_{13}}{g_{33}} ,$$
$$A_4 = -\log(g_{33}) \,, \qquad A_5 = -\frac{g_{34}}{g_{33}} , \qquad A_6 = -\frac{g_{32}}{g_{33}} .$$

Remark 4.2 Using pi-matrix techniques as expounded in [8, 11], we can find vector fields acting on functions of the variables $(A_1, \ldots, A_6)$ dual to the action of the Lie algebra on its universal enveloping algebra by multiplication from the right. This is the *right dual.* One finds for the Schrödinger algebra, with ∂_i denoting $\partial/\partial A_i$, $i = 1, \ldots, 6$:

$$
\begin{aligned}
M^* &= \partial_1 \,, \\
K^* &= \tfrac{1}{2} A_5^2 \partial_1 + \mathrm{e}^{2A_4} \partial_2 + A_5 \mathrm{e}^{A_4} \partial_3 + A_6 (\partial_4 + A_5 \partial_5 + A_6 \partial_6) \,, \\
G^* &= A_5 \partial_1 + \mathrm{e}^{A_4} \partial_3 + A_6 \partial_5 \,, \\
D^* &= \partial_4 + A_5 \partial_5 + 2 A_6 \partial_6 \,, \\
P_x^* &= \partial_5 \,, \\
P_t^* &= \partial_6 \,.
\end{aligned}
$$

Comparing with eqs. (1), suggests acting with these vector fields on functions of the form $\mathrm{e}^{mA_1}\,\mathrm{e}^{-dA_4}$ $f(A_5, A_6)$. Notice that the additional degrees of freedom corresponding to the variables A_2 and A_3 are ignored in this case. It is now seen that the physical realization is recovered from the right dual acting on functions of the form just indicated by the correspondence $x \leftrightarrow A_5$ and $t \leftrightarrow A_6$.

Referring to decomposition (3), we specialize variables, writing V_1, V_2, B_1, B_2 for A_2, A_3, A_6, A_5 respectively. Basic for our analysis is to establish the partial group law:

$$
\mathrm{e}^{B_1 P_t + B_2 P_x}\, \mathrm{e}^{V_1 K + V_2 G} = \,?
$$

We will get the required results using the matrix representation (4) noted above. The general elements of $\mathcal{P}$ and $\mathcal{L}$ are:

$$
B_1 P_t + B_2 P_x = \begin{pmatrix} 0 & B_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_1 & 0 & -B_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,
$$

$$
V_1 K + V_2 G = \begin{pmatrix} 0 & 0 & V_2 & 0 \\ 0 & 0 & V_1 & V_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .
$$

As the square of each of these matrices is zero, the exponential of each reduces to simply adding the identity. We find the matrix product

$$
\mathrm{e}^{B_1 P_t + B_2 P_x}\, \mathrm{e}^{V_1 K + V_2 G} = \begin{pmatrix} 1 & B_2 & V_2 + B_2\, V_1 & B_2\, V_2 \\ 0 & 1 & V_1 & V_2 \\ 0 & -B_1 & 1 - B_1\, V_1 & -B_1\, V_2 - B_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} .
$$

Applying Proposition 4.1 to the matrix found above yields

Proposition 4.3 *In coordinates of the second kind, we have the Leibniz formula,*

$$
\begin{aligned}
&g(0,0,0,0,B_2,B_1)\ g(0,V_1,V_2,0,0,0)\\
&= g\Bigg(\frac{1}{2}\,\frac{B_1\,{V_2}^2 + 2\,B_2\,V_2 + {B_2}^2 V_1}{1 - B_1\,V_1}, \frac{V_1}{1 - B_1\,V_1}, \frac{V_2 + B_2\,V_1}{1 - B_1\,V_1},\\
&\qquad -\log(1 - B_1\,V_1), \frac{B_1\,V_2 + B_2}{1 - B_1\,V_1}, \frac{B_1}{1 - B_1\,V_1}\Bigg).
\end{aligned}
$$

In general, a *Leibniz formula* is the group law for commuting the L operators past the R's, in analogy to the classical formula of Leibniz for derivatives.

4.3 Standard form of the Schrödinger algebra

Now we show the internal structure of the Schrödinger algebra ($n = 1$).

Remark 4.4 Note that we work in enveloping algebras throughout, so our calculations are based on relations in an associative algebra. In particular, we often use

$$[A, BC] = [A, B]C + B[A, C] \quad \text{and} \quad [A, B^2] = [A, B]B + B[A, B]\,. \tag{5}$$

Definition 4.5 Denote the basis for a standard Heisenberg-Weyl (HW) algebra, $\mathcal{H} = \mathrm{span}\{P, X, H\}$, satisfying

$$[P, X] = H, \quad [P, H] = [X, H] = 0\,.$$

A representation of HW-algebra such that H acts as the scalar m times the identity operator will be denoted as m-HW algebra.

Definition 4.6 Denote the basis for a standard sl(2) algebra, $\mathcal{K}$, by $\{L, R, \rho\}$, satisfying

$$[L, R] = \rho, \qquad [\rho, R] = 2R, \qquad [L, \rho] = 2L\,.$$

We write $\mathcal{K}\colon = \{L, R, \rho\}$.

The following Lemma is well-known. It follows readily from the equations in remark 4.4 (also see calculations below).

Lemma 4.7 *Given an m-*HW *algebra, setting*

$$L = \frac{1}{2m}\,P^2, \qquad \rho = \frac{1}{m}\,XP + \frac{1}{2}, \qquad R = \frac{1}{2m}\,X^2$$

yields a standard sl(2) *algebra.*

Now for our first main observation, which follows immediately from the commutation rules for the Schrödinger algebra.

Theorem 4.8 (HW form of the Schrödinger algebra) *Given an m-*HW *algebra, setting*

$$m = H\,, \quad K = \frac{1}{2m}\,X^2\,, \quad G = X\,, \quad D = \frac{1}{m}\,XP + \frac{1}{2}\,, \quad P_x = P\,, \quad P_t = \frac{1}{2m}\,P^2$$

yields a representation of $\mathcal{S}$.

And the main theorem, which gives the standard form.

Theorem 4.9 (Standard form of the Schrödinger algebra) *Any representation of the Schrödinger algebra* $\mathcal{S} = \mathrm{span}\{m, K, G, D, P_x, P_t\}$ *contains a standard* sl(2) *algebra* $\mathcal{K}_0 = \mathrm{span}\{L_0, R_0, \rho_0\}$ *such that, with the* m-HW *algebra* $\mathcal{H} = \mathrm{span}\{P_x, G, m\}$ *from the given representation of* $\mathcal{S}$*, the* sl(2) *subalgebra is of the form*

$$K = R_0 + \frac{1}{2m}\,G^2, \qquad D = \rho_0 + \frac{1}{m}\,GP_x + \frac{1}{2}, \qquad P_t = L_0 + \frac{1}{2m}\,P_x^2$$

where $\mathcal{K}_0$ *commutes with* $\mathcal{H}$.

Conversely, given any m-HW *representation, use it for* $\mathcal{H} := \{P_x, G, m\}$*. Now take any* sl(2) *algebra commuting with* $\mathcal{H}$*, and form the direct product with the standard* sl(2) *algebra constructed from* $\mathcal{H}$ *by the Lemma. Then this yields a representation of* $\mathcal{S}$.

P r o o f. The converse is clear by construction and our previous observations. What must be checked is that given a representation of $\mathcal{S}$, setting

$$R_0 = K - \frac{1}{2m}\,G^2\,, \qquad \rho_0 = D - \left(\frac{1}{m}\,GP_x + \frac{1}{2}\right)\,, \qquad L_0 = P_t - \frac{1}{2m}\,P_x^2$$

yields an sl(2) algebra that commutes with $\mathcal{H}$. From eq. (5), we have

$$[L_0, G] = [P_t, G] - \frac{1}{2m}\,\left[P_x^2, G\right] = P_x - P_x = 0$$

and similar relations for R_0 and ρ_0 show that $\mathcal{K}_0$ commutes with $\mathcal{H}$. Now, using remark 4.4, we note these relations

$$\begin{aligned}
\left[P_x^2, K\right] &= P_xG + GP_x = 2GP_x + m\,,\\
[GP_x, K] &= 0\cdot P_x + G\cdot G = G^2\,,\\
[P_t, GP_x] &= P_x^2\,.
\end{aligned}$$

Thus, using the fact that $[\mathcal{K}_0, \mathcal{H}] = 0$, we have

$$\begin{aligned}
[L_0, R_0] &= [L_0, K - \frac{1}{2m}\,G^2]\\
&= [P_t - \frac{1}{2m}\,P_x^2, K] + [L_0, -\frac{1}{2m}\,G^2]\\
&= D - \frac{1}{m}\,GPx - \frac{1}{2} = \rho_0
\end{aligned}$$

while

$$\begin{aligned}
[\rho_0, R_0] &= [\rho_0, K - \frac{1}{2m}\,G^2] = [\rho_0, K]\\
&= [D, K] - \frac{1}{m}\,[GP_x, K]\\
&= 2K - \frac{1}{m}\,G^2 = 2R_0
\end{aligned}$$

and

$$[L_0, \rho_0] = [P_t - \frac{1}{2m}\,P_x^2, \rho_0]$$

$$= [P_t, D] - [P_t, \frac{1}{m}\, GP_x]$$

$$= 2P_t - \frac{1}{m}\, P_x^2 = 2L_0$$

which completes the proof. □

Remark 4.10 The theorem, extended to include rotations, holds also for $n > 1$, where we use $\mathcal{K}_0$ spanned by

$$L_0 = \frac{1}{2m} \sum_i P_i^2\,, \qquad R_0 = \frac{1}{2m} \sum_i G_i^2\,, \qquad \rho_0 = \frac{1}{m} \sum_i G_i P_i + \frac{n}{2}$$

and for the rotations,

$$J_{0,ij} = J_{ij} - \frac{1}{m}\,(G_i P_j - G_j P_i)$$

with the J_0 rotations commuting with $\mathcal{H}$.

As an application of Theorem 4.8, consider the special realization, with scalar $M = m$ and x denoting multiplication by the variable x,

$$G = mx\,, \quad P_x = \frac{d}{dx}\,, \quad P_t = \frac{1}{2m}\,\frac{d^2}{dx^2}\,, \quad K = \frac{mx^2}{2}\,, \quad D = x\,\frac{d}{dx} + \frac{1}{2}\,. \tag{6}$$

In this realization, acting on the function identically equal to 1, we have $P_t 1 = P_x 1 = 0$, and $D1 = 1/2$. Applying a group element to the function 1, we find

$$g(A_1, A_2, A_3, A_4, A_5, A_6)\, 1 = \exp\left(A_1 m + A_2 \frac{mx^2}{2} + A_3 mx + A_4/2\right)\,.$$

Clearly, $f(x)1$ can be identified with the function $f(x)$ itself. Now apply the Leibniz formula, Proposition 4.3, to find

Corollary 4.11 *The differential realization of the Schrödinger algebra $\mathcal{S}_1$ has the following "partial group law"*

$$\begin{aligned}
&\exp\left(\frac{B_1}{2m}\,\frac{d^2}{dx^2} + B_2\,\frac{d}{dx}\right) \exp\left(V_1\,\frac{mx^2}{2} + V_2\, mx\right) \\
&\quad = \exp\left(\frac{V_1}{1 - B_1 V_1}\,\frac{mx^2}{2} + \frac{V_1 B_2 + V_2}{1 - B_1 V_1}\, mx\right) \\
&\qquad \times (1 - B_1 V_1)^{-1/2} \exp\left(\frac{m}{2}\,\frac{B_1\, {V_2}^2 + 2\, B_2\, V_2 + {B_2}^2 V_1}{1 - B_1\, V_1}\right)\,.
\end{aligned}$$

5 Canonical Appell systems for the Schrödinger algebra

Now we are ready to construct the representation space and basis – the canonical Appell system. To start, define a vacuum state Ω. The (commuting) elements K and G of $\mathcal{P}$ can be used to form basis elements

$$|jk\rangle = K^j G^k \Omega\,, \qquad j, k \geq 0$$

of a Fock space $\mathcal{F} = \mathrm{span}\{|jk\rangle\}$ on which K and G act as raising operators, while P_t and P_x act as lowering operators. That is, for constants m and c,

$$K\Omega = |10\rangle\,, \qquad G\Omega = |01\rangle\,,$$

$$P_t\Omega = 0\,, \qquad P_x\Omega = 0\,,$$
$$M\Omega = m|00\rangle\,, \qquad D\Omega = c|00\rangle\,.$$

Notation The standard form (cf. Theorem 4.9) gives $D = \rho_0 + (1/m)GP_x + 1/2$, which shows that $\rho_0\Omega = (c - 1/2)\Omega$. Hence in the following we denote $c - 1/2$ by $\dot{c}$.

5.1 Adjoint operators and Appell systems

The goal is to find an abelian subalgebra spanned by some selfadjoint operators acting on the representation space just constructed. Such a two-dimensional subalgebra can be obtained by an appropriate "turn" of the plane $\mathcal{P}$ in the Lie algebra, namely via the adjoint action of the group element formed by exponentiating P_t. The resulting plane, $\mathcal{P}_\beta$ say, is abelian and is spanned by

$$\begin{aligned} X_1 &= \mathrm{e}^{\beta P_t} K e^{-\beta P_t} = \exp(\mathrm{ad}\,\beta P_t)K = K + \beta D + \beta^2 P_t\,, \\ X_2 &= \mathrm{e}^{\beta P_t} G e^{-\beta P_t} = G + \beta P_x\,. \end{aligned} \tag{7}$$

Next we determine our canonical Appell systems. We want to compute $\exp(z_1X_1 + z_2X_2)\Omega$. Setting $V_1 = z_1$, $V_2 = z_2$, $B_1 = \beta$, and $B_2 = 0$ in Proposition 4.3 yields

$$\begin{aligned} \mathrm{e}^{z_1X_1}\mathrm{e}^{z_2X_2}\Omega &= \mathrm{e}^{\beta P_t}\mathrm{e}^{z_1K}\mathrm{e}^{z_2G}\mathrm{e}^{-\beta P_t}\Omega = \mathrm{e}^{\beta P_t}\mathrm{e}^{z_1K}\mathrm{e}^{z_2G}\Omega \\ &= \exp\left(\frac{z_1K}{1-\beta z_1}\right)\exp\left(\frac{z_2G}{1-\beta z_1}\right)(1-\beta z_1)^{-c}\exp\left(\frac{m}{2}\frac{\beta z_2^2}{1-\beta z_1}\right)\Omega\,. \end{aligned} \tag{8}$$

From eq. (8), we see K and G, our raising operators, while their respective adjoints P_t and P_x act as lowering operators on the basis $|jk\rangle = K^jG^k\Omega$. To get the generating function for the basis $|jk\rangle$, set in eq. (8)

$$v_1 = \frac{z_1}{1-\beta z_1}\,, \qquad v_2 = \frac{z_2}{1-\beta z_1}\,. \tag{9}$$

Substituting throughout, we have

Proposition 5.1 *The generating function for the canonical Appell system, $\{|jk\rangle = K^jG^k\Omega\}$ is*

$$\mathrm{e}^{v_1K}\mathrm{e}^{v_2G}\Omega = \exp\left(x_1\frac{v_1}{1+\beta v_1}\right)\exp\left(x_2\frac{v_2}{1+\beta v_1}\right)(1+\beta v_1)^{-c}\exp\left(-\frac{m\beta}{2}\frac{v_2^2}{1+\beta v_1}\right)$$

where we identify $X_1\Omega = x_1\cdot 1$ and $X_2\Omega = x_2\cdot 1$ in the realization as functions of x_1, x_2.

With $v_2 = 0$, we recognize the generating function for the Laguerre polynomials, while $v_1 = 0$ reduces to the generating function for Hermite polynomials. This corresponds to the results of Sect. 4 of [7].

From the exponentials $\exp(z_iX_i)$, eq. (8), we identify as operators $z_1 = \partial/\partial x_1$ and $z_2 = \partial/\partial x_2$. Using script notation for the v_i as operators, relations (9) take the form

$$\mathcal{V}_1 = \left(1 - \beta\frac{\partial}{\partial x_1}\right)^{-1}\frac{\partial}{\partial x_1}\,,$$
$$\mathcal{V}_2 = \left(1 - \beta\frac{\partial}{\partial x_2}\right)^{-1}\frac{\partial}{\partial x_2}\,.$$

To act on polynomials, expand $(1 - \beta\,\partial/\partial x_i)^{-1}$ in geometric series

$$(1 - \beta\,\partial/\partial x_i)^{-1} = \sum_{n\ge 0}\beta^n\left(\frac{\partial}{\partial x_i}\right)^n\,.$$

So we have both a $\mathcal{V}_1$-Appell system and a $\mathcal{V}_2$-Appell system as in Sect. 3. The Appell space decompositions are, for $\mathcal{V}_1$ and $\mathcal{V}_2$,

$$\mathcal{Z}_n^{(1)} = |\,\mathrm{poly}_n(K)\,\mathrm{poly}(G)\,\Omega\,\rangle\,,$$
$$\mathcal{Z}_n^{(2)} = |\,\mathrm{poly}_n(G)\,\mathrm{poly}(K)\,\Omega\,\rangle$$

respectively, where $\mathrm{poly}(\cdot)$, resp. $\mathrm{poly}_n(\cdot)$, denote arbitrary polynomials in the indicated variable, resp. of degree a most n in the variable. Now symmetries are generated by functions of $\partial/\partial x_1$ and $\partial/\partial x_2$. We will see explicit examples in Sect. 7.

5.2 Probability distributions

Now we shall consider some probabilistic observations. We introduce an inner product such that $K^* = \beta^2 P_t$ and $G^* = \beta P_x$. The X_i, which are formally symmetric, extend to self-adjoint operators on appropriate domains.

Expectation values are taken in the state Ω, i.e., for any operator Q,

$$\langle Q\rangle_\Omega = \langle \Omega, Q\Omega\rangle$$

where the normalization $\langle\Omega,\Omega\rangle = 1$ is understood.

From $P_t\Omega = P_x\Omega = 0$ follows that $\langle P_t\rangle_\Omega = \langle P_x\rangle_\Omega = 0$ and moving K and G across in the inner product, that $\langle K\rangle_\Omega = \langle G\rangle_\Omega = 0$ as well. Going back to eq. (8), take the inner product on the left with Ω. The exponential factors in K and G average to 1, yielding

$$\left\langle \mathrm{e}^{z_1X_1}\mathrm{e}^{z_2X_2}\right\rangle_\Omega = (1-\beta z_1)^{-c}\exp\left(\frac{m}{2}\frac{\beta z_2^2}{1-\beta z_1}\right)\,.$$

This result has an interesting probabilistic interpretation for positive values of β and c. Observe that the marginal distribution of X_1 (i.e., for $z_2 = 0$) is gamma distribution, while the marginal distribution of X_2 (now $z_1 = 0$) is Gaussian. Note, however, that these are not independent random variables.

To recover the joint distribution of X_1, X_2, let us first recall some probability integrals (Fourier transforms):

$$\int_{-\infty}^{\infty} \mathrm{e}^{i\xi y}\mathrm{e}^{-\lambda y}y^{t-1}\lambda^t\,\theta(y)\,dy/\Gamma(t) = (1-i\xi/\lambda)^{-t}\,,\quad \text{for } t>0\,,$$
$$\int_{-\infty}^{\infty} \mathrm{e}^{-i\eta u}\mathrm{e}^{-u^2/(2v)}\,du = \sqrt{2\pi v}\,\mathrm{e}^{-\eta^2 v/2}\,,\quad \text{for } v>0$$

where $\theta(x)$ denotes the usual Heaviside function, $\theta(x) = 1$ if $x \geq 0$, zero otherwise. Replacing z_1, z_2 by iz_1, iz_2 respectively and taking inverse Fourier transforms, we have

Proposition 5.2 *The joint density $p(x_0, x_2)$ of the random variables X_1, X_2 is given by*

$$p(x_0,x_2) = \mathrm{e}^{-x_0/\beta}\mathrm{e}^{-x_2^2/(2m\beta)}x_0^{\dot c-1}\beta^{-\dot c}\,\theta(x_0)\,\frac{dx_0\,dx_2}{\Gamma(\dot c)\sqrt{2\pi m\beta}}$$

for $c, \beta > 0$, where $\dot c = c - 1/2$ and $x_0 = x_1 - x_2^2/(2m)$.

The result says that the marginal distribution of X_2 is Gaussian with mean 0 and variance $2m\beta$. Conditional on X_2, X_1 is gamma with parameters $1/\beta$ and $c - 1/2$ taking values in the interval $(x_2^2/(2m), \infty)$. In the special case $c = 1/2$, i.e., $\dot c = 0$, the gamma density collapses to a delta function: $\delta(x_1 - x_2^2/(2m))$.

6 Leibniz function and orthogonal basis

Once the Leibniz formula for the Lie algebra $\mathcal{S}_1$ is known (Proposition 4.3), we can proceed to define coherent states, find the Leibniz function – inner product of coherent states – and show that we have a Hilbert space with self-adjoint commuting operators $X_1 = P_t + D + K$ and $X_2 = G + P_x$ (here the β in eqs. (7) is set equal to 1). We recover the raising and lowering operators as elements of the Lie algebra acting on the Hilbert space with basis consisting of the canonical Appell system.

The two-parameter family of coherent states is defined as

$$\psi_V = \psi_{V_1,V_2} = \mathrm{e}^{V_1 K}\mathrm{e}^{V_2 G}\Omega\,.$$

Using Proposition 4.3, we see

Proposition 6.1 *With $K^* = P_t$ and $G^* = P_x$, the Leibniz function is*

$$\Upsilon_{BV} = (1 - B_1V_1)^{-c}\exp\left(\frac{m}{2}\frac{B_1V_2^2 + 2B_2V_2 + B_2^2V_1}{1 - B_1V_1}\right)\,.$$

Proof. Use Proposition 4.3 in the relation

$$\Upsilon_{BV} = \langle\psi_B, \psi_V\rangle = \left\langle\Omega, \mathrm{e}^{B_2P_x}\mathrm{e}^{B_1P_t}\,\mathrm{e}^{V_1K}\mathrm{e}^{V_2G}\Omega\right\rangle\,.$$

□

Note that the Leibniz function is symmetric in B and V, which is equivalent to the inner product being symmetric, and thus the Hilbert space being well-defined.

It is remarkable that the Lie algebra can be reconstructed from the Leibniz function Υ_{BV}. The idea is that differentiation Υ_{BV} with respect to V_1 brings down K acting on ψ_V, while differentiation with respect to B_1 brings down a K acting on ψ_B which moves across the inner product as P_t acting on ψ_V. Similarly for G and P_x. We thus introduce creation operators $\mathcal{R}_i$, and annihilation (velocity) operators $\mathcal{V}_i$, satisfying $[\mathcal{V}_i, \mathcal{R}_j] = \delta_{ij}\mathrm{I}$. Expressing the Lie algebra in terms of these operators is the *boson realization.* We thus identify $K = \mathcal{R}_1$, $G = \mathcal{R}_2$. Note, however, that $\mathcal{V}_1$ is not the adjoint of $\mathcal{R}_1$, nor $\mathcal{V}_2$ that of $\mathcal{R}_2$. In fact, our goal is to determine the boson realization of P_t and P_x, the respective adjoints.

Here is a method to find the boson realization. First, one determines the partial differential equations for $\Upsilon = \Upsilon_{BV}$:

$$\begin{aligned}
\frac{\partial\Upsilon}{\partial B_1} &= V_1^2\frac{\partial\Upsilon}{\partial V_1} + V_1V_2\frac{\partial\Upsilon}{\partial V_2} + cV_1\Upsilon + \tfrac{m}{2}V_2^2\Upsilon\,,\\
\frac{\partial\Upsilon}{\partial B_2} &= V_1\frac{\partial\Upsilon}{\partial V_2} + mV_2\Upsilon\,.
\end{aligned}$$

Then, one interprets each multiplication by V_i as the operator $\mathcal{V}_i$ and each differentiation by V_i as the operator $\mathcal{R}_i$. This gives the following action of the operators P_x and P_t on polynomial functions of K and G:

$$P_x = m\mathcal{V}_2 + \mathcal{R}_2\mathcal{V}_1\,,\qquad P_t = c\mathcal{V}_1 + \mathcal{R}_1\mathcal{V}_1^2 + \tfrac{m}{2}\mathcal{V}_2^2 + \mathcal{R}_2\mathcal{V}_1\mathcal{V}_2\,.$$

This means that P_x acts on $|jk\rangle = \mathcal{R}_1^j\mathcal{R}_2^k|00\rangle$ as follows

$$P_x|jk\rangle = m\,k|j, k-1\rangle + j|j-1, k+1\rangle$$

and P_t does similarly. The element D is recovered via

$$D = [P_t, K] = \left[c\mathcal{V}_1 + \mathcal{R}_1\mathcal{V}_1^2 + \tfrac{m}{2}\mathcal{V}_2^2 + \mathcal{R}_2\mathcal{V}_1\mathcal{V}_2, \mathcal{R}_1\right] = c\,\mathrm{I} + 2\mathcal{R}_1\mathcal{V}_1 + \mathcal{R}_2\mathcal{V}_2\,.$$

To tie in with more usual notation for bosons, let us set $a_1 = \mathcal{V}_1$, $a_2 = \mathcal{V}_2$, $a_1^+ = \mathcal{R}_1$, and $a_2^+ = \mathcal{R}_2$, with the proviso that a_1 and a_1^+, e.g., are not in fact adjoint to each other. We may formulate the boson realization thus:

$$\begin{aligned}
M &= m\,\mathrm{I}\,,\\
K &= a_1^+\,,\\
G &= a_2^+\,,\\
D &= c\,\mathrm{I} + 2a_1^+a_1 + a_2^+a_2\,,\\
P_x &= m\,a_2 + a_2^+a_1\,,\\
P_t &= c\,a_1 + a_1^+a_1^2 + \tfrac{m}{2}a_2^2 + a_2^+a_1a_2\,.
\end{aligned}$$

To recover the physical realization, eqs. (1), first use the natural involution on the algebra $K \leftrightarrow P_t$, $G \leftrightarrow P_x$, then note the substitutions $a_1 \to t$, $a_2 \to x$, $a_1^+ \to \partial_t$, $a_2^+ \to \partial_x$, $c \to -d$. Then reorder so that the derivatives are on the right (Wick ordering).

Summarizing the action of the boson realization, we have

Theorem 6.2 *The representation of the Schrödinger algebra on the Fock space $\mathcal{F}$ with basis $|j,k\rangle = K^jG^k\Omega$ is given by*

$$\begin{aligned}
K\,|j,k\rangle &= |j{+}1,k\rangle\,,\\
G\,|j,k\rangle &= |j,k{+}1\rangle\,,\\
P_x\,|j,k\rangle &= mk\,|j,k{-}1\rangle + j\,|j{-}1,k{+}1\rangle\,,\\
P_t\,|j,k\rangle &= j(c{+}k{+}j{-}1)\,|j{-}1,k\rangle + \tfrac{m}{2}k(k{-}1)\,|j,k{-}2\rangle\,,\\
D\,|j,k\rangle &= (c{+}2j{+}k)\,|j,k\rangle\,,\\
M\,|j,k\rangle &= m\,|j,k\rangle\,.
\end{aligned}$$

Corollary 6.3 *In the above representation, the Schrödinger operator $\mathbf{S} = P_t - P_x^2/(2m)$ is represented by $\mathbf{S} = \dot{c}\mathcal{V}_1 + \mathcal{R}_o\mathcal{V}_1^2$, or $\dot{c}\,a_1 + a_0^+a_1^2$, i.e.,*

$$\mathbf{S}\,|j,k\rangle = j(\dot{c} + j - 1)|j-1,k\rangle + \tfrac{1}{2m}j(j-1)\,|j-2,k+2\rangle$$

where we define $\mathcal{R}_0 = \mathcal{R}_1 - \mathcal{R}_2^2/(2m)$, $a_0^+ = a_1^+ - (a_2^+)^2/(2m)$, cf. Theorem 4.9.

A very important feature of the Leibniz function Υ_{BV} is that it is the generating function for the inner products of the elements of the basis. Indeed, expanding the exponentials defining the coherent states yields

$$\Upsilon_{BV} = \sum_{j,k,j',k'} \langle jk|j'k'\rangle\,\frac{B_1^jB_2^kV_1^{j'}V_2^{k'}}{j!k!j'!k'!}\ .$$

For an orthogonal basis, a necessary and sufficient condition is that this must be a function only of the pair products B_1V_1 and B_2V_2. We proceed to find an orthogonal basis.

Lemma 6.4 *The Leibniz function can be expressed as*

$$\Upsilon_{BV} = (1 - B_1V_1)^{-\dot{c}}\,\exp\left(\frac{B_1}{2m}\,\frac{\partial^2}{\partial B_2^2} + \frac{V_1}{2m}\,\frac{\partial^2}{\partial V_2^2}\right)\,\mathrm{e}^{mB_2V_2}$$

with $\dot{c} = c - 1/2$.

P r o o f. In the formulation of Corollary 4.11 first set $B_2 = 0$. Then use the special realization as in eq. (6) with $x = B_2$. As in Corollary 4.11

$$\exp\left(\frac{B_1}{2m}\frac{d^2}{dB_2^2}\right)\exp\left(V_1\frac{mB_2^2}{2}+V_2\,mB_2\right)$$
$$=\exp\left(\frac{V_1}{1-B_1V_1}\frac{mB_2^2}{2}+\frac{V_2}{1-B_1V_1}\,mB_2\right)(1-B_1V_1)^{-1/2}\exp\left(\frac{m}{2}\,\frac{B_1\,V_2^2}{1-B_1\,V_1}\right)$$

which combines to yield Υ_{BV} up to the factor $(1-B_1V_1)^{-\dot c}$. Now observe that

$$\mathrm{e}^{mB_2V_2+(m/2)V_1B_2^2}=\exp\left(\frac{V_1}{2m}\,\frac{\partial^2}{\partial V_2^2}\right)\mathrm{e}^{mB_2V_2}$$

where on $\exp(mB_2V_2)$, $\partial/\partial V_2$ acts simply as multiplication by mB_2. Combining with the above observations yields the result. □

Now for the main result, expressing the basis in terms of $R_0 = K - G^2/(2m)$, cf. Corollary 6.3.

Theorem 6.5 *The set* $|rs\rangle = R_0^rG^s\Omega$, $r, s \ge 0$, *forms an orthogonal basis with squared norms*

$$\langle rs|rs\rangle = (\dot c)_r\, r!s!m^s$$

where $(\dot c)_r = \dot c(\dot c+1)\dots(\dot c+r-1)$.

P r o o f. From Lemma 6.4,

$$\begin{aligned}(1-B_1V_1)^{-\dot c}\,\mathrm{e}^{mB_2V_2}&=\exp\left(-\frac{B_1}{2m}\,\frac{\partial^2}{\partial B_2^2}-\frac{V_1}{2m}\,\frac{\partial^2}{\partial V_2^2}\right)\left\langle \mathrm{e}^{B_1K+B_2G}\Omega,\mathrm{e}^{V_1K+V_2G}\Omega\right\rangle\\&=\left\langle \mathrm{e}^{B_1(K-G^2/(2m))+B_2G)}\Omega,\mathrm{e}^{V_1(K-G^2/(2m))+V_2G}\Omega\right\rangle\\&=\left\langle \mathrm{e}^{B_1R_0+B_2G}\Omega,\mathrm{e}^{V_1R_0+V_2G}\Omega\right\rangle\,.\end{aligned}$$

Now we have the generating function for the inner products $\langle rs|r's'\rangle$ depending only on the pair products B_1V_1, B_2V_2. Hence orthogonality. Expanding the left-hand side of the equation yields the squared norms. □

Similarly, we have for the canonical Appell system,

Proposition 6.6 *Let* $X_0 = X_1 - X_2^2/(2m)$, *with the identification* $X_0\Omega = x_0\cdot 1$. *Then*

$$\mathrm{e}^{v_0R_0}\mathrm{e}^{v_2G}\Omega=\exp\left(x_0\frac{v_0}{1+\beta v_0}\right)(1+\beta v_0)^{-\dot c}\exp\left(x_2v_2-\beta m\,v_2^2/2\right)\,.$$

P r o o f. First substitute v_0 for v_1 in Proposition 5.1. And observe that

$$\mathrm{e}^{v_0R_0}\mathrm{e}^{v_2G}\Omega=\exp\left(-\frac{v_0}{2m}\,\frac{\partial^2}{\partial v_2^2}\right)\mathrm{e}^{v_0K}\mathrm{e}^{v_2G}\Omega\,.$$

Now use the special realization, eq. (6), taking $x = v_2$ in Corollary 4.11, with

$$B_1=-v_0\,,\qquad B_2=0\,,\qquad V_1=-\frac{\beta}{1+\beta v_0}\,,\qquad V_2=\frac{x_2/m}{1+\beta v_0}\,.$$

After substituting accordingly and simplifying, one finds the stated result. □

Note that now the system decouples into Laguerre polynomials in the variable x_0 and Hermite polynomials in the variable x_2.

7 Appell systems and evolution equations

The generating function, hence basis functions, for a canonical Appell system will, in general, satisfy a system of evolution equations of the form $\dfrac{\partial u}{\partial \tau_i} = H_i u$ with the H_i commuting *natural Hamiltonians* for the system (cf. [5] where their Hamiltonians are our X's). These correspond to the abelian subalgebra of Cartan elements, acting as scalars on the vacuum state. For the Schrödinger algebra these are M and D. The corresponding variables are m and c. In this context, then, set $\tau_1 = c$, $\tau_2 = m$.

In Proposition 5.1, denote the generating function by u. Observe that

$$\left(1 - \beta\frac{\partial}{\partial x_1}\right) u = \frac{1}{1+\beta v_1} u\,,$$

$$\frac{\partial u}{\partial x_2} = \frac{v_2}{1+\beta v_1} u\,.$$

Thus, substituting $c = \tau_1$, $m = \tau_2$ we find the evolution equations

$$\frac{\partial u}{\partial \tau_1} = -\log(1+\beta v_1)\, u = H_1 u = \log\left(1 - \beta\frac{\partial}{\partial x_1}\right) u\,,$$

$$\frac{\partial u}{\partial \tau_2} = -\frac{\beta}{2}\frac{v_2^2}{1+\beta v_1} u \qquad = H_2 u = -\frac{\beta}{2}\left(1 - \beta\frac{\partial}{\partial x_1}\right)^{-1} \frac{\partial^2 u}{\partial x_2^2}\,.$$

Comparing with Proposition 5.2 shows that the first equation corresponds to a Lévy process with time-parameter τ_1, a gamma process, i.e., convolution powers of an exponential distribution. The second equation, a modified Fokker-Planck equation, corresponds to a coupled Brownian motion.

The system is decoupled in the variables x_0, x_2, as indicated in Proposition 6.6. Proceeding similarly as above, substitute $\dot{c} = \tau_1$, $m = \tau_2$ in the generating function expressed in the variables x_0, x_2. Now we have the evolution equations

$$\frac{\partial u}{\partial \tau_1} = H_1 u = \log\left(1 - \beta\frac{\partial}{\partial x_0}\right) u\,,$$

$$\frac{\partial u}{\partial \tau_2} = H_2 u = -\frac{\beta}{2}\frac{\partial^2 u}{\partial x_2^2}\,.$$

From which it is clear that the system corresponds to independent Lévy processes: a gamma process and a Brownian motion process.

8 Conclusion

The methods indicated here that have been developed for Lie algebras/Lie groups show nicely the structure of the Schrödinger algebra and the connections between the Schrödinger algebra and quantum probability. It is interesting to see that physically interesting representations/realizations are natural consequences of our approaches: the duality approach via pi-matrices and the approach via the Leibniz function using generalized coherent states/Berezin quantization.

The methods used here extend to the $n = d + 1$-dimensional Schrödinger algebra. Of special interest is the rôle played by the subgroup of rotations.

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