Berezin Quantization of the Schrödinger Algebra

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Abstract. We examine the Schrödinger algebra in the framework of Berezin quantization. First, the Heisenberg-Weyl and sl(2) algebras are studied. Then the Berezin representation of the Schrödinger algebra is computed. In fact, the sl(2) piece of the Schrödinger algebra can be decoupled from the Heisenberg component. This is accomplished using a special realization of the sl(2) component that is built from the Heisenberg piece as the quadratic elements in the Heisenberg-Weyl enveloping algebra. The structure of the Schrödinger algebra is revealed in a lucid way by the form of the Berezin representation.

Keywords: Lie algebras, Schrödinger algebra, Heisenberg-Weyl algebra, sl(2), coherent states, Berezin representation, Leibniz function.

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1 Introduction

The Schrödinger algebra is a Lie algebra that has attracted since its introduction [12, 14] considerable interest in mathematical physics and its applications (see, e.g., [1, 2, 3, 7, 8]).

In [9] we have investigated the semidirect product structure of the Schrödinger algebra and showed how it leads to representations in a Fock space realized in terms of canonical Appell systems. This includes a classification of the representations and construction of the Hilbert space of functions on which certain commuting elements act as self-adjoint operators. Some associated evolution equations have been considered as well. The notion of generalized coherent states is exploited extensively there.

Here we shall take a somewhat different point of view and study the Schrödinger algebra using the method of "Berezin quantization," which we understand from the rather broad point of view as developed from the original work of Berezin by Perelomov and others, see [4, 5, 6, 15]. Again, the generalized coherent states play an essential rôle.

Here is a description of the contents of this paper. Section 2 presents the Schrödinger algebra. Section 3 contains the basics of our formulation of Berezin's theory. The Berezin representation for each of the Heisenberg-Weyl and sl(2) algebras is presented in §4. The Berezin quantization of the Schrödinger algebra constitutes §5. Concluding remarks and some further lines for research are given in §6.

2 Schrödinger algebra

The ($n = 1$, centrally-extended) Schrödinger algebra $\mathcal{S}$ is spanned by the following elements

M	mass
K	special conformal transformation
G	Galilei boost
D	dilation
P_x	spatial translation
P_t	time translation

(see, e.g., [8] for details) which satisfy the following commutation relations given here in the form of a multiplication table

$$\begin{array}{c} \\ M \\ K \\ G \\ D \\ P_x \\ P_t \end{array} \begin{array}{c} \begin{array}{cccccc} M & K & G & D & P_x & P_t \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\,K & -G & -D \\ 0 & 0 & 0 & -G & -M & -P_x \\ 0 & 2\,K & G & 0 & -P_x & -2\,P_t \\ 0 & G & M & P_x & 0 & 0 \\ 0 & D & P_x & 2\,P_t & 0 & 0 \end{pmatrix} \end{array}$$

(thus, e.g., $[D, K] = 2K$). The Schrödinger algebra can be viewed as a semidirect product

$$\mathcal{S} \cong \mathcal{H} \oplus_s sl(2)$$

of two subalgebras: a Heisenberg-Weyl subalgebra $\mathcal{H} = \text{span}\,\{M, G, P_x\}$, and $sl(2) = \text{span}\,\{K, D, P_t\}$.

This fundamental feature is considered in some detail in [9].

3 Cartan decomposition and Berezin theory

Our approach to Berezin quantization [4, 5, 6] is based on the exposition in [15] that uses the notion of generalized coherent state as a group element acting on an appropriate vacuum state, but, like in [13], goes beyond the "symmetric space paradigm." For more on the calculational tools used, the reader may consult [10].

Consider a Lie algebra $\mathfrak{g}$ that admits a splitting

$$\mathfrak{g} = \mathcal{L} \oplus \mathcal{K} \oplus \mathcal{P} \tag{1}$$

where $\mathcal{R}$ and $\mathcal{L}$ are two abelian subalgebras of the same dimension n, such that they generate the whole algebra: $\mathfrak{g} = \text{gen}\,\{\mathcal{L}, \mathcal{P}\}$.

Remark 3.1 An important case of such a structure is the *Cartan decomposition* for symmetric Lie algebras with $\mathcal{L}$ and $\mathcal{P}$ satisfying $[\mathcal{L}, \mathcal{P}] \subseteq \mathcal{K}$, $[\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}$, and $[\mathcal{K}, \mathcal{L}] \subseteq \mathcal{L}$. This precise structure, however, does not exist for the Schrödinger algebra, which does not correspond to a classical symmetric space. In fact there are two possibilities for a generalized "Cartan decomposition." One satisfies the appropriate commutation properties, but does not obey $\dim \mathcal{P} = \dim \mathcal{L}$, see [9]. The other—a different variation on the standard Cartan decomposition—will be used in the present context.

Let $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{K}$ have bases $\{R_j\}$, $\{L_j\}$, and $\{\rho_A\}_{1 \le A \le m}$, respectively. A typical element $X \in \mathfrak{g}$ is of the form

$$X = v'_j R_j + u'_A \rho_A + w'_j L_j \tag{2}$$

for some $(2n+m)$-tuple (v', u', w'). A group element can be obtained either by direct exponentiation of X, or by composing exponentials corresponding to the factorization into subgroups according to the decomposition of the Lie algebra. Thus

$$e^X = \exp(v_j R_j) \left(\prod_A \exp(u_{\dot{A}} \rho_{\dot{A}}) \right) \exp(w_j L_j) \tag{3}$$

We use the convention of summation over repeated indices, unless the indices are dotted (the dot indicating no summation over A). Clearly, the coordinates (v, u, w) versus (v', u', w') are mutually dependent as they represent in (3) the same group element.

Now, we construct a representation space $\mathcal{H}$ for the enveloping algebra of $\mathfrak{g}$ as a Fock space spanned by basis elements

$$|k_1, k_2, \ldots, k_n\rangle = R_1^{k_1} \cdots R_n^{k_n} \Omega \tag{4}$$

where Ω is a vacuum state. The action of the algebra elements on the vacuum state is defined thus

(i) $\hat{R}_j \Omega = R_j \Omega$

(ii) $\hat{L}_j \Omega = 0$

(iii) $\hat{\rho}_A \Omega = \tau_A \Omega$

where τ_A are constants. Next, we equip $\mathcal{H}$ with a symmetric scalar product in some number field, such that the ladder operators are mutually adjoint:

$$\hat{R}_i^* = \hat{L}_i$$

The adjoint map for other elements is determined by the commutation rules. We shall always consider the vacuum state normalized, $\langle \Omega | \Omega \rangle = 1$.

In an important special case, one assumes that only one element of $\mathcal{K}$, say ρ_0, acts on Ω as a nonzero constant τ. Hence the group element specified by equation (3) acts on Ω as follows

$$e^X \Omega = e^{\tau\, u} \exp(v_j R_j) \Omega \tag{5}$$

The system possesses two types of lowering and raising operators (ladder operators): *algebraic* and *combinatorial.* The *algebraic lowering* and *raising* operators are defined simply by concatenation within the enveloping algebra of $\mathfrak{g}$ followed by acting on Ω, that is

$$\begin{aligned} \hat{R}_j \psi &= R_j \psi \\ \hat{L}_j \psi &= L_j \psi \end{aligned} \tag{6}$$

for any linear combination ψ of basis elements (4). The "hat" can be thus omitted without causing confusion. The *combinatorial raising operators*, $\mathcal{R}_j$, and *combinatorial lowering operators*, $\mathcal{V}_j$, are defined to act on the basis (by definition) as follows

$$\begin{aligned}\mathcal{R}_j\,|k_1,k_2,\ldots,k_n\rangle &= |k_1,k_2,\ldots,k_j+1,\ldots k_n\rangle\\ \mathcal{V}_j\,|k_1,k_2,\ldots,k_n\rangle &= k_j\,|k_1,k_2,\ldots,k_j-1,\ldots k_n\rangle\end{aligned}$$

("off-diagonal matrices").

Next, the idea will be to express the *algebraic* ladder operators, $\hat{L}_j$, $\hat{R}_j$ (and hence the basis for $\mathfrak{g}$), in terms of the *combinatorial* ladder operators $\mathcal{R}_j$ and $\mathcal{V}_j$.

It is clear that the algebraic raising operators are represented directly by the $\mathcal{R}$'s, namely $\hat{R}_j = \mathcal{R}_j$. But the combinatorial lowering operators do not necessarily correspond to elements of $\mathfrak{g}$. To find the representation we shall use the coherent states.

Definition 3.2 The system of coherent states $\mathcal{C}$ is the image of the subgroup generated by the (abelian) subalgebra $\mathcal{R} \subset \mathfrak{g}$ in the Hilbert space $\mathcal{H}$ constructed above, namely $\mathcal{C} = \exp \mathcal{R} \cdot \Omega$ with the typical element

$$|v\rangle = \exp(v_j R_j)\Omega$$

The coherent states form a manifold $\mathcal{C}$ parametrized by the elements of $\mathcal{R}$, or, equivalently, by coordinates $v = (v_1, \ldots, v_n)$.

Observation 3.3 When restricted to coherent states, $\mathcal{R}_j$ acts as differentiation $\partial/\partial v_j$, while $\mathcal{V}_j$ acts as multiplication by v_j:

$$\begin{aligned}\mathcal{R}_j &= \partial/\partial v_j \qquad (\text{on } \mathcal{C})\\ \mathcal{V}_j &= v_j\cdot\end{aligned}$$

We shall use this property to determine the action of any operator defined as a (formal) operator function $f(\mathcal{R}, \mathcal{V})$, with all $\mathcal{V}$'s to the right of any $\mathcal{R}_j$, by (1) moving all $\mathcal{R}$'s to the right of all $\mathcal{V}$'s in the formula f, yielding the operator $\check{f}(\mathcal{V}, \mathcal{R})$, and then (2) replacing $\mathcal{V}_j \to v_j$ and $\mathcal{R}_j \to \partial/\partial v_j$. Note that this is a formal Fourier transform combined with the *Wick ordering.* The Berezin transform extends this by taking the inner product with a coherent state $|w\rangle$.

The following notion will be used frequently.

Definition 3.4 The *Leibniz function* is a map $\mathcal{C} \times \mathcal{C} \to \mathbf{C}$ defined as the inner product of the coherent states:

$$\Upsilon_{wv} = \langle w|v\rangle$$

for any v, w parametrizing $\mathcal{C}$.

The Leibniz function can be explicitly calculated for a particular Lie algebra as a scalar function symmetric with respect to v and w. The calculations are based on the adjoint structure: we start with

$$\langle w|v\rangle = \langle \exp(w_jR_j)\Omega \,|\, \exp(v_jR_j)\Omega\rangle = \langle \Omega \,|\, \exp(w_jL_j)\exp(v_jR_j)\Omega\rangle$$

and then use a formula for commuting a typical group element generated by L's past a typical group element generated by R's, that is, generally

$$e^L\, e^R = e^r e^k e^l \tag{7}$$

where $R \in \mathcal{R}$ and $L \in \mathcal{L}$ are general elements, while $r \in \mathcal{R}$ and $l \in \mathcal{L}$ and $k \in \mathcal{K}$ are functions of the coordinates of R and L and need to be computed in each particular case from the commutation relations. (See §4 below for explicit examples.) The relation (7) is called in the following the *Leibniz formula.*

Definition 3.5 *The* coherent state representation (Berezin transform) *is defined for an operator Q as*

$$\langle Q\rangle_{wv} = \frac{\langle w|Q|v\rangle}{\langle w|v\rangle}\,.$$

The Berezin transforms of the algebraic ladder operators can be expressed in terms of derivatives of the Leibniz function, namely

$$\begin{aligned}
\langle \hat{R}_j\rangle_{wv} &= \Upsilon^{-1}\frac{\partial}{\partial v_j}\Upsilon = \frac{\partial(\log\Upsilon)}{\partial v_j}\\
\langle \hat{L}_j\rangle_{wv} &= \Upsilon^{-1}\frac{\partial}{\partial w_j}\Upsilon = \frac{\partial(\log\Upsilon)}{\partial w_j}
\end{aligned}$$

(using the fact that L_j is adjoint to R_j). The right-hand sides are functions of v and w. Hence, by "eliminating w's," one may find a system of first-order partial differential equations satisfied by Υ, say

$$\frac{\partial\Upsilon}{\partial w_j} = \check{f}_j(v, \frac{\partial}{\partial v})\,\Upsilon$$

for some operator functions $\check{f}_j$. We shall call this a system of *defining partial differential equations.* As indicated in the discussion above, it gives the answer to our question of the representation of $\hat{L}_j$, namely

$$\hat{L}_j = f_j(\mathcal{R}, \mathcal{V})$$

Remark 3.6 Note that the converse holds as well: if we have $\hat{L}_j$ expressed via $\mathcal{R}$ and $\mathcal{V}$, then Υ satisfies the corresponding partial differential equation. In some cases, this can be used to find Υ.

Also, note that in the case of symmetric spaces, $\log\Upsilon$ is the *Kähler potential.*

One goal is to identify in our representation a set of n mutually commuting self-adjoint operators X_j — observables — which provide physical or probabilistic interpretations for certain elements of the Lie algebra. For instance, they generate a unitary group, $\exp(i\sum_j s_j X_j)$, for $s = (s_1, \ldots, s_n) \in \mathbf{R}^n$ where $i = \sqrt{-1}$. The scalar function defined by

$$\phi(s) = \langle\Omega| \exp(i \sum_j s_j X_j)|\Omega\rangle \tag{8}$$

is positive-definite, so, by *Bochner's Theorem*, $\phi(s)$ is the Fourier transform of a positive measure, which is, in fact, the joint spectral density of the observables $(X_1, \ldots, X_n)$.

4 Berezin quantization in action

Now we shall see how these general ideas are executed in the case of the Heisenberg-Weyl and sl(2) algebras. How these results appear combined in the case of the Schrödinger algebra will be shown in the following Section.

4.1 The Heisenberg-Weyl algebra

First we define a standard form of the Heisenberg-Weyl algebra.

Definition 4.1 The standard basis for a Heisenberg-Weyl (HW) algebra $\mathcal{H} \cong \operatorname{span}\{P, X, H\}$ satisfies

$$[P, X] = H, \quad [P, H] = [X, H] = 0$$

Given a scalar $m > 0$, an m-HW algebra denotes a representation where H acts as the scalar m times the identity operator.

The Leibniz formula for the m-HW algebra is

$$e^{wP}e^{vX} = e^{vX}e^{m\,wv}e^{wP}$$

(known in the literature as the Weyl formula and essential in quantum mechanics). The Hilbert space (Fock space), has basis $|n\rangle = X^n\Omega$ with rules $P\Omega = 0$, $H\Omega = m\Omega$. ¿From the equation above and the relation $P = X^*$, the Leibniz function can be easily calculated:

$$\Upsilon_{wv} = \langle e^{wX}\Omega|e^{vX}\Omega\rangle = \exp(mwv)$$

where we assume a normalized vacuum state. Then, the following *defining partial differential equation*

$$\frac{\partial\Upsilon}{\partial w} = mv\Upsilon$$

suggests how the algebra basis can be expressed in terms of the combinatorial ladder operators $\mathcal{R}$, $\mathcal{V}$, namely

$$\hat{X} = \mathcal{R}, \quad \hat{H} = m, \quad \hat{P} = m\mathcal{V}$$

Remark 4.2 Note that in this case, we could actually find the action of P directly using the adjoint action of the group:

$$\begin{aligned} P\,|v\rangle &= Pe^{vX}\Omega = e^{vX}e^{-vX}Pe^{vX}\Omega \\ &= e^{vX}e^{-v\,\mathrm{ad}\,X}P\Omega = e^{vX}(P+mv)\Omega \\ &= mve^{vX}\Omega = m\mathcal{V}\,|v\rangle \end{aligned}$$

For the Berezin transforms, first we have $\langle X\rangle_{wv} = \Upsilon^{-1}\partial\Upsilon/\partial v = mw$ so, from $P = X^*$, exchanging $w \leftrightarrow v$, we immediately have $\langle P\rangle_{wv} = mv$. Summarizing,

Proposition 4.3 *For the m-HW algebra, the Berezin representation is*

$$\langle X\rangle_{wv} = mw, \quad \langle H\rangle_{wv} = m, \quad \langle P\rangle_{wv} = mv$$

Note that the Berezin representation of the operator $X_1 = X + P$ is $\langle X_1\rangle_{wv} = m(w+v)$, which, being symmetric in w and v, is formally self-adjoint.

Recall the exponential formula

$$\exp(aP_x + bG) = e^{bG}\exp(mab/2)e^{aP_x} \tag{9}$$

which can be readily verified by differentiation and the adjoint action. Then we have the function $\phi(s)$, as in equation (8),

$$\phi(s) = \langle\Omega|\exp(isX_1)|\Omega\rangle = e^{-s^2m/2}$$

which is the well-known Fourier transform of a normal distribution with density function $\exp(-x^2/(2m))/\sqrt{2\pi m}$. Thus, we interpret X_1 as a Gaussian random variable with variance m.

4.2 The sl(2) algebra

Now we proceed similarly with the algebra sl(2).

Definition 4.4 The standard basis of the sl(2) algebra $\mathcal{A} \cong$ span $\{L, R, \rho\}$ satisfies

$$[L, R] = \rho, \quad [\rho, R] = 2R, \quad [L, \rho] = 2L$$

In this basis, the sl(2) Leibniz formula is

$$e^{wL}e^{vR} = \exp(\frac{v}{1-wv}R)(1-wv)^{-\rho}\exp(\frac{w}{1-wv}L) \tag{10}$$

This can be computed using differential equations, as in [10], or using 2×2 matrices, cf. [11].

Our Hilbert space has basis $|n\rangle = R^n\Omega$ with the rules $L\Omega = 0$, and $\rho\Omega = c\Omega$ for a constant c. With $L = R^*$, the Leibniz function follows easily from the Leibniz formula:

$$\Upsilon_{wv} = \langle e^{wR}\Omega | e^{vR}\Omega\rangle = (1 - wv)^{-c}$$

The Leibniz function satisfies the partial differential equation

$$\frac{\partial \Upsilon}{\partial w} = cv\Upsilon + v^2\,\frac{\partial \Upsilon}{\partial v}$$

from which one can read the following representation of the algebra in terms of the combinatorial operators $\mathcal{R}$, $\mathcal{V}$:

$$\hat{R} = \mathcal{R}, \qquad \hat{L} = c\mathcal{V} + \mathcal{R}\mathcal{V}^2$$

(As in the HW case, we alternatively can find the action of L via the adjoint action of the group.)
To find $\hat{\rho}$, we calculate $[L, R]$ to get:

$$\hat{\rho} = [\hat{L}, \hat{R}] = [c\mathcal{V} + \mathcal{R}\mathcal{V}^2, \mathcal{R}] = c + 2\mathcal{R}\mathcal{V}$$

For the Berezin transforms, we have

Proposition 4.5 *The Berezin representation of the sl(2) algebra is*

$$\langle R\rangle_{wv} = \frac{cw}{1 - wv}, \quad \langle \rho\rangle_{wv} = c\,\frac{1 + wv}{1 - wv}, \quad \langle L\rangle_{wv} = \frac{cv}{1 - wv}$$

Proof: The transform for R comes directly by taking the logarithmic derivative of Υ with respect to v. Then the result for L follows as it is adjoint to R. For ρ, convert $\hat{\rho} = c + 2\mathcal{R}\mathcal{V}$ to $c + 2v\partial \log \Upsilon/\partial v$ to find the stated result. □

Now consider $X_2 = R + \rho + L$. We have

$$\langle X_2\rangle_{wv} = c\,\frac{(1 + w)(1 + v)}{1 - wv}$$

which is symmetric in w and v, showing that X_2 defines a formally self-adjoint operator.

5 Berezin quantization of the Schrödinger algebra

We now will see how the results of the previous section relate to the Schrödinger algebra $\mathcal{S}$.

First we find the Leibniz formula and the Leibniz function for $\mathcal{S}$, and then the Berezin representations of its basis elements. Next we will identify two (essentially) self-adjoint operators acting in the Hilbert space for $\mathcal{S}$. Further investigation of the Berezin representation will lead to a structure theorem for the Schrödinger algebra.

We consider the following decomposition of the Schrödinger algebra (braces represent spanning)

$$\mathcal{S} = \underbrace{\{\, P_x, P_t \,\}}_{\mathcal{L}} \oplus \underbrace{\{\, M, \rho \,\}}_{\mathcal{K}} \oplus \underbrace{\{\, K, G \,\}}_{\mathcal{P}}$$

Thus, in terms of the notation of Section 3, we associate $R_1 = K$, $R_2 = G$, $L_1 = P_t$ and $L_2 = P_x$.

The Hilbert space $\mathcal{H}$ is a Fock space defined as

$$\mathcal{H} = \overline{\text{span}}\,\{\, |n_1, n_2\rangle \equiv K^{n_1} G^{n_2} \Omega \,\}$$

the bar indicating closure of the linear span, and the algebra elements act on the vacuum state Ω as follows

$$P_x\Omega = P_t\Omega = 0, \qquad D\Omega = c\Omega, \qquad M\Omega = m \cdot \Omega$$

so that on any element of $\mathcal{H}$, M acts as multiplication by the scalar m. We assume that $\mathcal{H}$ is equipped with an inner product such that $K^* = P_t$ and $G^* = P_x$. The consistency of this — existence of such an inner product — follows from the symmetry of the Leibniz function calculated below.

This construction makes the following two operators essentially self-adjoint:

$$\begin{aligned} X_1 &= P_t + D + K \\ X_2 &= G + P_x \end{aligned}$$

The system of coherent states $\mathcal{C}$ is defined as a two-parameter manifold in $\mathcal{H}$ with typical element

$$|v\rangle = |v_1, v_2\rangle = e^{v_1 K} e^{v_2 G} \Omega$$

Lemma 5.1 *The Leibniz formula for the Schrödinger algebra, reduced by acting on the vacuum state, is*

$$e^{w_1 P_t + w_2 P_x} e^{v_1 K + v_2 G} \Omega = (1 - w_1 v_1)^{-c} \exp(\frac{m}{2}\, \tilde{q}(w,v)) \exp(\tilde{v}_1 K + (\tilde{v}_2 + w_2 \tilde{v}_1) G)$$

where

$$\begin{aligned} \tilde{v}_i &= v_i/(1 - w_1 v_1), \qquad i = 1, 2 \\ \tilde{q}(w,v) &= \frac{m}{2}\, \frac{w_1 v_2^2 + 2 w_2 v_2 + w_2^2 v_1}{1 - w_1 v_1} \end{aligned}$$

Proof: We take several steps to pull the P-factors across all terms.

1. Apply the Leibniz formula for sl(2):

$$e^{w_1 P_t} e^{v_1 K} = e^{\tilde{v}_1 K} (1 - w_1 v_1)^{-D} e^{\tilde{w}_1 P_t}$$

   with $\tilde{w}_1 = w_1/(1 - w_1 v_1)$.

2. Recall the HW formula, equation (9)

$$\exp(aP_x + bG) = e^{bG} \exp(mab/2) e^{aP_x}$$

   Now the adjoint action gives

$$e^{\tilde{w}_1 P_t} G e^{-\tilde{w}_1 P_t} = G + \tilde{w}_1 P_x$$

   and hence from the above HW formula,

$$e^{\tilde{w}_1 P_t} e^{v_2 G} \Omega = \exp(\frac{m}{2} \tilde{w}_1 v_2^2) e^{v_2 G} \Omega$$

3. Next, since D acts a dilation on G,

$$(1 - w_1 v_1)^{-D} e^{v_2 G} \Omega = (1 - w_1 v_1)^{-c} e^{\tilde{v}_2 G} \Omega$$

4. Now for the P_x-factor, the adjoint action gives

$$e^{w_2 P_x} K e^{-w_2 P_x} = K + w_2 G + m w_2^2/2$$

   and exponentiating,

$$e^{w_2 P_x} e^{\tilde{v}_1 K} = e^{\tilde{v}_1 K} e^{\tilde{v}_1 w_2 G} \exp(m \tilde{v}_1 w_2^2/2) e^{w_2 P_x}$$

5. And the HW Leibniz formula is the last step:

$$e^{w_2 P_x} e^{\tilde{v}_2 G} = e^{\tilde{v}_2 G} \exp(m w_2 \tilde{v}_2) e^{w_2 P_x}$$

Combining the factors involving m, K, and G yields the result stated. □

This formula now yields the Leibniz function.

Proposition 5.2 *The Leibniz function for the Schrödinger algebra is:*

$$\Upsilon_{wv} = (1 - w_1 v_1)^{-c} \exp\Big(\frac{m}{2} \frac{w_1 v_2^2 + 2 w_2 v_2 + w_2^2 v_1}{1 - w_1 v_1}\Big)$$

Proof: Apply the Leibniz formula in

$$\Upsilon_{wv} = \langle w|v\rangle = \langle\Omega|e^{w_2 P_x} e^{w_1 P_t}\, e^{v_1 K} e^{v_2 G}\Omega\rangle$$

and use the fact that appropriate elements of $\mathcal{L}$ and $\mathcal{P}$ are mutually adjoint, specifically, $K^* = P_t$ and $G^* = P_x$. □

Clearly, Υ is symmetric in w and v which shows the symmetry property of the inner product.

Now for the Berezin transforms of the Lie algebra elements, including the self-adjoint X-operators. The above proposition implies the following system of partial differential equations:

$$\begin{aligned}
\frac{\partial\Upsilon}{\partial w_1} &= v_1^2\,\frac{\partial\Upsilon}{\partial v_1} + v_1 v_2\frac{\partial\Upsilon}{\partial v_2} + c v_1\Upsilon + \frac{m}{2}\,v_2^2\Upsilon \\
\frac{\partial\Upsilon}{\partial w_2} &= v_1\,\frac{\partial\Upsilon}{\partial v_2} + m v_2\Upsilon
\end{aligned}$$

from which we can infer the hat-representation of our Lie algebra

$$\begin{aligned}
\hat{P}_t &= c\mathcal{V}_1 + \tfrac{m}{2}\,\mathcal{V}_2^2 + (\mathcal{R}_1\mathcal{V}_1 + \mathcal{R}_2\mathcal{V}_2)\mathcal{V}_1 \\
\hat{K} &= \mathcal{R}_1 \\
\hat{D} &= c + 2\mathcal{R}_1\mathcal{V}_1 + \mathcal{R}_2\mathcal{V}_2 \\
\hat{P}_x &= m\mathcal{V}_2 + \mathcal{R}_2\mathcal{V}_1 \\
\hat{G} &= \mathcal{R}_2 \\
\hat{M} &= m
\end{aligned}$$

To get $\hat{D}$, we used the commutation rule $D = [P_t, K]$. As a result of these calculations, the following Berezin representation emerges

$$\begin{aligned}
\langle P_t\rangle_{wv} &= c\,\frac{v_1}{1 - w_1 v_1} + \frac{m}{2}\left(\frac{w_2 v_1 + v_2}{1 - w_1 v_1}\right)^2 \\
\langle P_x\rangle_{wv} &= m\,\frac{w_2 v_1 + v_2}{1 - w_1 v_1} \\
\langle K\rangle_{wv} &= c\,\frac{w_1}{1 - w_1 v_1} + \frac{m}{2}\left(\frac{w_2 + w_1 v_2}{1 - w_1 v_1}\right)^2 \\
\langle G\rangle_{wv} &= m\,\frac{w_2 + w_1 v_2}{1 - w_1 v_1} \\
\langle D\rangle_{wv} &= c\,\frac{1 + w_1 v_1}{1 - w_1 v_1} + m\,\frac{(w_2 v_1 + v_2)(w_2 + w_1 v_2)}{(1 - w_1 v_1)^2} \\
\langle X_1\rangle_{wv} &= c\,\frac{(1 + w_1)(1 + v_1)}{1 - w_1 v_1} + m\left(\frac{w_2 + v_2 + w_1 v_2 + w_2 v_1}{1 - w_1 v_1}\right)^2 \\
\langle X_2\rangle_{wv} &= m\,\frac{v_2 + w_2 + w_1 v_2 + w_2 v_1}{1 - w_1 v_1}
\end{aligned}$$

where the transforms for X_i are found by adding the appropriate results. In this form it is clear that indeed $K^* = P_t$, $G^* = P_x$, $D^* = D$ and $X_i^* = X_i$, which verifies the validity of the Hilbert space constructed. The case $m = 0$ recovers the sl(2) case, cf. §4.2. However, the $c = 0$ case is interesting, as it is unlike either of the Heisenberg-Weyl or the sl(2) cases. The Berezin representation of X_2 shows that it is not simply an independent Gaussian, which would look just like m times the sum $w_2 + v_2$, cf. the operator X_1 in §4.1.

The above formulas suggest that one should perform the following subtractions:

$$P_t - P_x^2/(2m), \quad K - G^2/(2m), \quad D - GP_x/m - \tfrac{1}{2}$$

the $\frac{1}{2}$ arises naturally as will be seen shortly. We start with

Proposition 5.3 *In the hat-representation, define $\mathcal{R}_0 = \mathcal{R}_1 - \mathcal{R}_2^2/(2m)$. Then*

$$\begin{aligned} \hat{P}_t - \hat{P}_x^2/(2m) &= (c - \tfrac{1}{2})\,\mathcal{V}_1 + \mathcal{R}_0\mathcal{V}_1^2 \\ \hat{K} - \hat{G}^2/(2m) &= \mathcal{R}_0 \\ \hat{D} - \hat{G}\hat{P}_x/m - \tfrac{1}{2} &= (c - \tfrac{1}{2}) + 2\mathcal{R}_0\mathcal{V}_1 \end{aligned}$$

Proof: These follow readily from the commutation relations for the $\mathcal{R}$ and $\mathcal{V}$ operators. □

Now, we find

Theorem 5.4 *For the Schrödinger algebra, we have*

$$\begin{aligned} \langle P_t - \frac{1}{2m}\,P_x^2\rangle_{wv} &= (c - \tfrac{1}{2})\,\frac{v_1}{1 - w_1v_1} \\ \langle K - \frac{1}{2m}\,G^2\rangle_{wv} &= (c - \tfrac{1}{2})\,\frac{w_1}{1 - w_1v_1} \\ \langle D - \frac{1}{m}\,GP_x - \tfrac{1}{2}\rangle_{wv} &= (c - \tfrac{1}{2})\,\frac{1 + w_1v_1}{1 - w_1v_1} \end{aligned}$$

Consequently, $L_0 = P_t - \frac{1}{2m}\,P_x^2$, $R_0 = K - \frac{1}{2m}\,G^2$, $\rho_0 = D - \frac{1}{m}\,GP_x - \frac{1}{2}$ form a standard basis of an sl(2) algebra.

Proof: Use the hat-representation from Proposition 5.3 in the dual form acting on the Leibniz function. Setting L_0, R_0, and ρ_0 as in the statement of the Theorem,

$$\begin{aligned} \check{L}_0 &= (c - \tfrac{1}{2})\,v_1 + v_1^2\left(\frac{\partial}{\partial v_1} - \frac{1}{2m}\,\frac{\partial^2}{\partial v_2^2}\right) \\ \check{R}_0 &= \left(\frac{\partial}{\partial v_1} - \frac{1}{2m}\,\frac{\partial^2}{\partial v_2^2}\right) \\ \check{\rho}_0 &= (c - \tfrac{1}{2}) + 2v_1\left(\frac{\partial}{\partial v_1} - \frac{1}{2m}\,\frac{\partial^2}{\partial v_2^2}\right) \end{aligned}$$

and compute accordingly. □

As to the main structure of the Schrödinger algebra, we have

Theorem 5.5 *The elements L_0, R_0, ρ_0 defined in Theorem 5.4 commute with the Heisenberg-Weyl subalgebra generated by P_x, G, m.*

Proof: Use the hat-representation found in Proposition 5.3. With $\hat{P}_x = m\mathcal{V}_2 + \mathcal{R}_2\mathcal{V}_1$ and $\hat{G} = \mathcal{R}_2$, it is readily checked that each of the sl(2) operators commutes with P_x and G. □

6 Concluding remarks

- For the case $n > 1$, an interesting approach would be to study representations induced from (the Lie algebra of) the Euclidean group. On the other hand, the rotation subgroup splits off by subtracting operators of the form $G_iP_j - G_jP_i$ from the J_{ij} rotation operators (cf. [9]). But dealing with the representations induced from the rotation subgroup requires some more detailed work (cf., "intrinsic" subalgebras in Hecht [13]).

- Thanks to the decoupling structure, extending our approach to the q-Schrödinger algebra looks quite reasonable.

- Finding the finite-dimensional representations is another project to be considered.

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