Disentangling a triangle

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Additional problems and hints

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i. The formulas derived hold for obtuse triangle as well.

   [Hint: *Take $\triangle RST$ with excircle of diameter 1 and corresponding angles ρ, σ, τ. Assume that $\sigma \geq \pi/2$. Look at Figures 5 and 6: rename $\triangle RST$ as $\triangle A\bar{C}B$. Note that $\rho = \frac{\pi}{2} - \beta$, $\sigma = \alpha + \beta$, $\tau = \frac{\pi}{2} - \alpha$. Using reduction formulas for sine and cosine, establish the validity of all the formulas received so far for $\triangle RST$.*]

ii. The area of a triangle is $(ABC) = \frac{1}{2} \sin\alpha \sin\beta \sin\gamma$.

iii. $4 \sin\alpha \sin\beta \sin\gamma < \pi$.

   [Hint: *Two copies of $\triangle ABC$, the triangle itself and its three parts, each of them with vertex in H, do not fill the circumcircle of radius $\frac{1}{2}$.*]

iv. The power of H in the circumcircle of $\triangle ABC$ (product of any two parts of a chord passing through the point) is $2 \cos\alpha \cos\beta \cos\gamma$.

   [Hint: *Take the product of length of the ear and the sum lengths of the stalk and the root.*]

v. Distance of the antipodal point of C from AB equals to the root h_C.

   [Hint: *Segments $\bar{C}D$ and AB in Figure 11 are parallel.*]

vi. The circle inscribed in the orthic triangle has radius $\cos A \cos B \cos C$ and center H.

   [Hint: *By Proposition 8 (iii), H is the center of the incircle of $\triangle A_F B_F C_F$. Thus the distance from H to the side $A_F C_F$ is the root of the altitude from B in the triangle $A_F B C_F$, so it is $\cos\alpha \cos\gamma$ scaled by the factor $\cos\beta$, see Figure 10.*]

vii. Orthocenters of $\triangle A C_F B_F$, $\triangle B A_F C_F$ and $\triangle C B_F A_F$ lie on sides of the circum-orthic triangle.

   [Hint: *The distance between H and $A_F B_F$ is the same as between the parallel sides $A_F B_F$ and $\bar{A}\bar{B}$. It is the length of root of the altitude from C in $\triangle A_F B_F C$.*]

viii. The distance $u = |OH|$ satisfies $u^2 = 1/4 - 2 \cos\alpha \cos\beta \cos\gamma$.

   [Hint: *Take the chord through H and O. One of its segments measures $1/2 + u$, the other $1/2 - u$.*]

ix. For $\alpha+\beta+\gamma=\pi$ one has $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1$.

[Hint: *In* $\triangle ABH$ *obtain* $\sin\gamma$*, length of* AB*, using the theorem of cosines; two other sides measure* $\cos\alpha$ *and* $\cos\beta$ *and form the angle* $\pi-\gamma$*.*]

x. The heights of a triangle obey the inequality

$$\frac{1}{|h_A|} + \frac{1}{|h_B|} > \frac{1}{|h_C|}\,.$$

[Hint: *Divide the triangular inequality* $\sin\alpha + \sin\beta > \sin\gamma$ *by the product of the sines.*]

xi. The inradius r of $\triangle ABC$ satisfies

$$\frac{1}{r} = \frac{1}{|h_A|} + \frac{1}{|h_B|} + \frac{1}{|h_C|}$$

[Hint: *Use the relation* $r(\sin\alpha+\sin\beta+\sin\gamma) = \sin\alpha\sin\beta\sin\gamma$ *.*]

xii. The centers of circumcircles of $\triangle ABH$, $\triangle BCH$, $\triangle CAH$ form the triangle $O_{AB}O_{BC}O_{CA}$ congruent to the triangle ABC and they are interchanged by the half-turn around N.

[Hint: *The distance from* O *to the side* AB *is* $\frac{1}{2}|HC|$*, see the proof of Proposition 12. Thus, the reflection of* O *in* AB *yields* O_{AB} *such that* $OO_{AB}\|HC$ *and* $|OO_{AB}| = |HC|$*; in the new triangle* O *becomes the orthocenter.*]