Disentangling a Triangle

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1. INTRODUCTION. In his *Almagest*, Ptolemy inscribes triangles in a unit circle, a circle with diameter $d = 1$ (see [**5**], pp. 90–92). This way the length of each side (now chord) represents the value of the trigonometric function **sine** of the opposite angle. A similar geometric interpretation of the **cosine** function is possible.

In Figure 1 we present Ptolemy's famous sketch, juxtaposed with a "dual" sketch that shows the values of the cosines as segments of the altitudes (see Proposition 1).

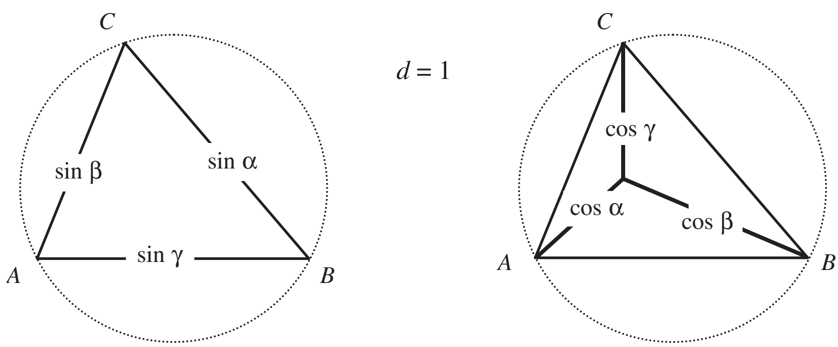


Figure 1. Ptolemy's design and its dual.

This observation—which seems to be absent in all the presentations of trigonometry known to us—proves to be a convenient tool for bringing order into the garden of trigonometric identities. As an example, see Figure 2 for ways to visualize products of sines and of cosines; they would perhaps please Napier in his experiments with such products that eventually led him to the concept of logarithm.

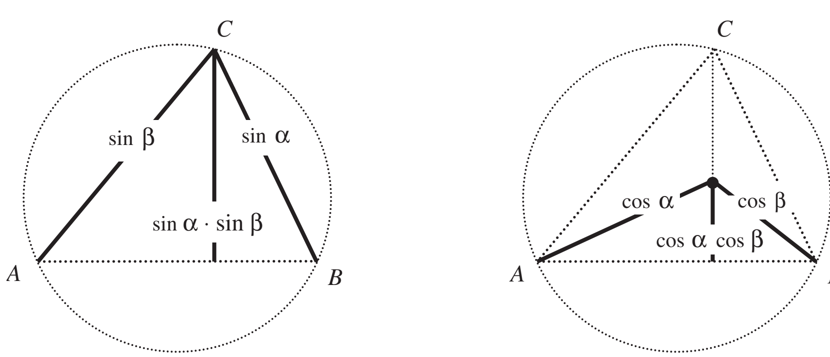


Figure 2. Products of trigonometric functions.

In the following we present some new proofs and visualizations. Along the way, we also prove some standard facts to make the "sine-cosine" duality transparent, and to maintain completeness, with the aim of providing a guide for a geometric path through trigonometry. Among new items are visualizations shown on the right in Figures 1, 2, 14, and 15. Also, a simple proof of the *Nine-Point Circle Theorem* stands among the benefits we present.

Throughout these notes we use the term **unit circle** for one with unit diameter, $d = 2r = 1$. We deal with **acute** triangles only; the case of right and obtuse triangles is handled in Section 6. It is reduced there to the former one by construction of the **unit orthotetrad**, the set of three vertices and the orthocenter.

2. A GOOD LOOK AT THE ALTITUDES. We use the standard notation for a triangle: letters A, B, and C denote the vertices, α, β, and γ the corresponding angles. As all the results remain valid under permutation of vertices, we state properties and conduct reasoning for a particular choice of the elements.

The altitude (see Figure 3) from vertex C is denoted by h_C; its **foot** (the point lying on the side AB) by C_F. The point on the other end of the chord containing h_C is denoted by $\overline{C}$.

The orthocenter H (its existence remains yet to be proven) divides each altitude into two segments: its **ear** CH extends from the orthocenter to the vertex and its **stem** HC_F goes from its foot to the orthocenter. The segment $C_F\overline{C}$ that goes beyond the triangle and reaches the circle is called a **root**. Recall that the midpoint of the ear is known as an **Euler point**.

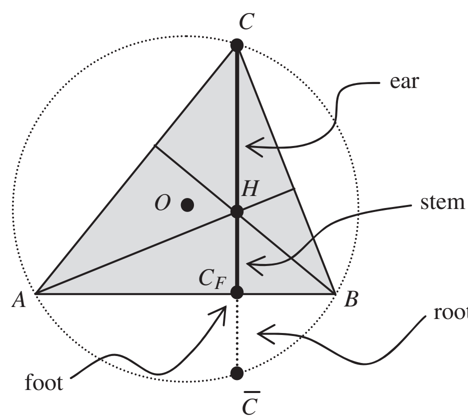


Figure 3. The anatomy of an altitude.

Let us begin with proofs of the facts illustrated in Figure 1.

Proposition 1. *Let a triangle ABC be inscribed in a unit-diameter circle.*

(i) [Ptolemy's theorem] *The triangle sides represent sines:* $|AB| = \sin\gamma$.

(ii) [It's dual] *The ears represent cosines:* $|CH| = \cos\gamma$.

(iii) [Existence of orthocenter] *The three altitudes concur in one point H.*

Proof. Both angles $\gamma = \angle ACB$ and $\angle AC_1B$ in the left-hand side of Figure 4 are based on the same chord AB, so they are equal. The point C_1 is chosen so that AC_1 forms a diameter and $\angle ABC_1$ is a right angle. The definition of sine implies (i).

To prove (ii), consider altitude $h_B = BB_F$. As $\triangle AC_1C$ is a right triangle, both CC_1 and BB_F are perpendicular to AC, and therefore mutually parallel. Thus forming parallelogram BC_1CH we get $|CH| = |C_1B| = \cos\gamma$ (see Figure 4).

To prove (iii), repeat the process of (ii) using A instead of B to see that H is the point where h_C and h_A meet. ■

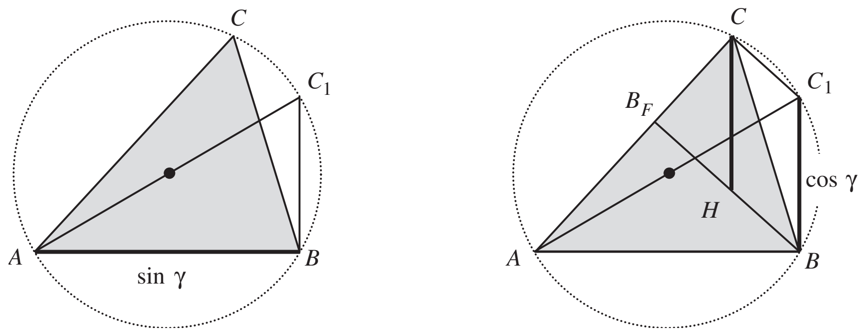


Figure 4. For $\sin\gamma$, follow Ptolemy. For $\cos\gamma$, shift it to vertex C.

Scaling the figure by the factor $d \in \mathbb{R}_+$ yields these basic trigonometric facts:

Corollary 2. *If $\triangle ABC$ is inscribed in a circle of diameter d, then*

(i) [The Law of Sines] $$\frac{|AB|}{\sin\gamma} = \frac{|BC|}{\sin\alpha} = \frac{|AC|}{\sin\beta} = d.$$

(ii) [The New Law of Cosines] $$\frac{|AH|}{\cos\alpha} = \frac{|BH|}{\cos\beta} = \frac{|CH|}{\cos\gamma} = d.$$

Proof. Inscribe $\triangle ABC$ in a circle of some diameter d and dilate with respect to its center to a unit circle to get a "Ptolomean" triangle. The corollary may be viewed as a reformulation of Thales' theorem. ■

The altitudes cut a triangle into six triangles. The resulting angles are shown in Figure 5, where φ^* denotes the angle complementary to φ ($\varphi^* = \frac{\pi}{2} - \varphi$).

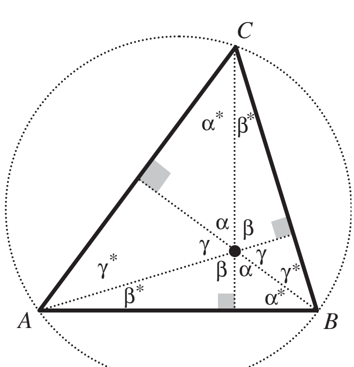


Figure 5. Angles.

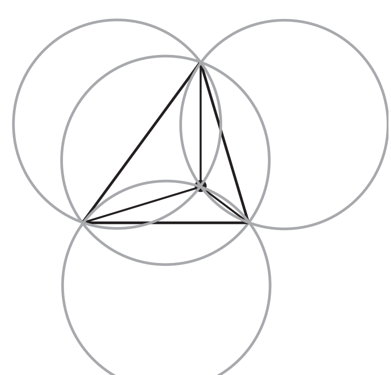


Figure 6. Four unit circles.

Proposition 3. *A circle through any three of the four points A, B, C, H has the same diameter (Figure 6).*

Proof. Use the Law of Sines to find the diameter d_{AHB} of the circle circumscribed on $\triangle ABH$. The angle opposite AB is $\alpha + \beta = \pi - \gamma$, so the diameter is

$$d_{AHB} = \frac{|AB|}{\sin(\pi - \gamma)} = \frac{\sin\gamma}{\sin\gamma} = 1. \qquad \blacksquare$$

Proposition 4. *The stem and root of each altitude are of equal length.*

Proof. Both angles $\angle ABC$ and $\angle A\overline{C}C$ are based on the same chord AC, thus $\angle A\overline{C}C = \angle ABC = \beta$. On the other hand $\angle AH\overline{C} = \beta$ (see Figure 5). An analogous argument gives $\angle H\overline{C}B = \angle BH\overline{C} = \alpha$. So, $\triangle A\overline{C}B$ and $\triangle AHB$ are congruent. $\blacksquare$

Corollary 5. *The reflections of the orthocenter H through the sides of $\triangle ABC$ lie on the excircle of the triangle.*

This picture complements Proposition 3 on the four unit excircles. Each of the four points is the orthocenter of the triangle formed by the other three. It should also be obvious that, by Proposition 3, any pair of these circles form mutual mirror reflections through their common chord.

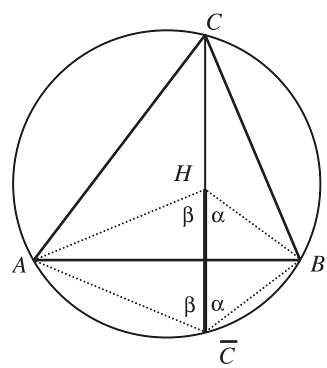


Figure 7. The stem and the root are equal.

Remark. Frère Gabriel-Marie gives the name of **orthocentric group** to such configurations, attributing the nomenclature to an article by de Longchamps from 1891 (see [**2**], p. 1076) and their study to Carnot (see [**2**], p. 142). For related material, see Chapter 2 of [**3**].

Let us now formulate the results displayed in Figure 2.

Proposition 6. *In a triangle ABC inscribed in a unit circle:*

(i) *The length of an altitude is the product of the sines of the angles opposite the altitude:* $|h_C| = \sin\alpha \cdot \sin\beta$.

(ii) *The length of a stem is the product of the cosines of the angles opposite the altitude that contain the stem:* $|HC_F| = \cos\alpha \cdot \cos\beta$.

Proof. (i) $\dfrac{|h_C|}{\sin\beta} = \sin\alpha$. (ii) $\dfrac{|HC_F|}{\cos\beta} = \cos\alpha$. $\blacksquare$

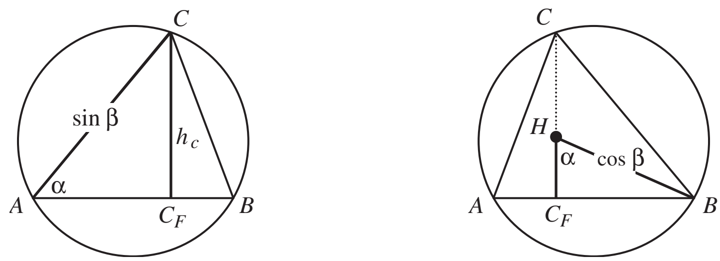


Figure 8. Proof of Corollary 7: Use definitions of sine and cosine.

As a bonus we get from Figure 8 geometric representations of mixed products of sines and cosines:

Corollary 7. *In a triangle ABC inscribed in a unit circle, the foot of an altitude h_C divides the side AB into segments of lengths:*

$$|AC_F| = \cos\alpha \sin\beta \quad \textit{and} \quad |BC_F| = \cos\beta \sin\alpha.$$

Proof. If suffices to check Figure 8 for $\dfrac{|AC_F|}{\sin\beta} = \cos\alpha$ and for $\dfrac{|BC_F|}{\cos\beta} = \sin\alpha$. ∎

3. REPLICAS. The **orthic** triangle $A_F B_F C_F$ is spanned by the feet of the altitudes. Interestingly, by removing it from $\triangle ABC$ we get three smaller copies of the original triangle (see Figures 9 and 10). To see this, we determine the angles of the triangles as in Figure 9 right.

There is a useful—although somewhat neglected—notion, present in older geometry textbooks. If a line cuts BC in N and AC in M forming $\angle NMC = \alpha$, it is **parallel** to AB. If $\angle NMC = \beta$, the line is called **antiparallel** to AB (see [**1**], p. 169).

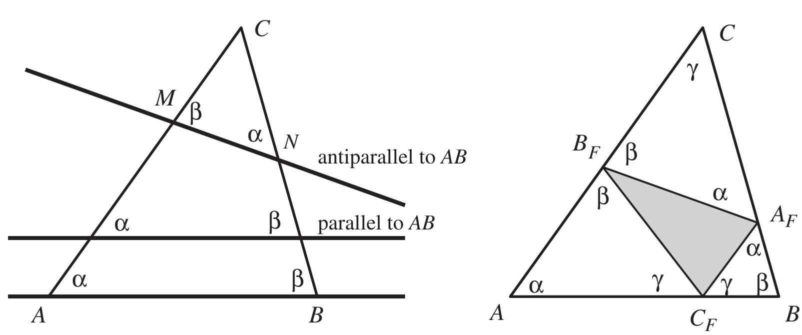


Figure 9. Orthic $\triangle A_F B_F C_F$ is built from antiparallels.

Proposition 8. *Let $\triangle A_F B_F C_F$ be the orthic triangle of $\triangle ABC$. Then:*

(i) *The sides of the orthic triangle are antiparallel to the sides of $\triangle ABC$.*

(ii) *The angle of that triangle at the vertex B_F is $\pi - 2\beta$.*

(iii) *The altitudes of $\triangle ABC$ are bisectors of the angles of $\triangle A_F B_F C_F$.*

Proof. (i) The quadrilateral $A_F C B_F H$ (see Figure 10, on the left) has two right angles at the opposite vertices A_F and B_F—so, it is cyclic. Now, $\angle A_F HC = \angle A_F B_F C$ as these angles are subtended by the same chord $A_F C$ in the circumscribed circle. The first of them (see Figure 5) is β. Therefore the segment $A_F B_F$ lies on a line antiparallel to AB.

(ii) Similar reasoning applied to the quadrilateral $B_F AC_F H$ leads to the conclusion that $\angle C_F B_F A = \beta$, so $\angle A_F B_F C_F = \pi - 2\beta$.

(iii) Since $\angle A_F B_F C = \angle C_F B_F A$ and HB_F is perpendicular to AC, line HB_F bisects $\angle A_F B_F C_F$. ■

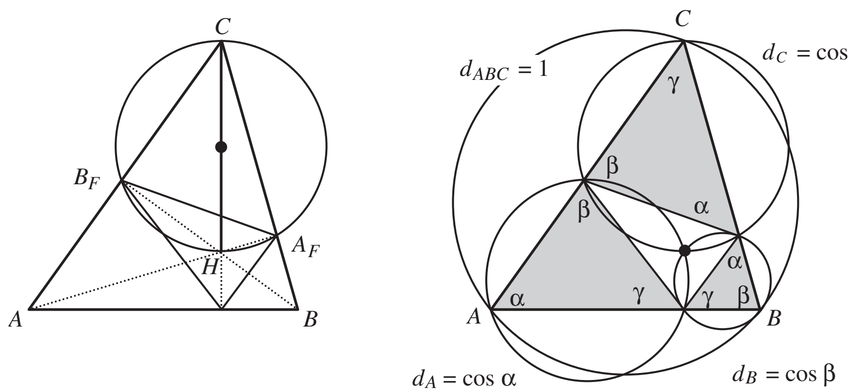


Figure 10. New circumcircles.

Thus, by removing $\triangle A_F B_F C_F$ from $\triangle ABC$ one gets three copies of the original triangle, with circumcircles that have diameters $d_A = \cos\alpha$, $d_B = \cos\beta$, and $d_C = \cos\gamma$. Automatically, we obtain geometric interpretations of multiple products of at most two sines and arbitrarily many factors of cosines, as segments in a fractal-style nested family of ever smaller copies of $\triangle ABC$.

Notice that the centers of the three circles are at the midpoints of the ears of the altitudes of $\triangle ABC$.

Our sketch (Figure 11) also contains differences of angles:

Proposition 9. *Let CD be a diameter of the circle circumscribing $\triangle ABC$.*

(i) *The diameter CD is orthogonal to $A_F B_F$.*

(ii) *If $\beta > \alpha$ then $\angle \overline{C}CD = \beta - \alpha$.*

Proof. (i) By Proposition 8 we have $\angle B_F A_F C = \alpha$. On the other hand, $\angle BDC$ and $\angle BAC$ are subtended by the same chord BC, so $\angle BDC = \alpha$. As CD is a diameter, $\triangle BCD$ is a right triangle and $\angle BCD = \alpha^*$, so that $\angle A_F C_1 C = \pi/2$.

(ii) In right triangles $B_F C_1 C$ and $BC_F C$ there are angles equal to β at B_F and at B, respectively (see Figure 11). Thus, $\angle B_F CC_1 = \angle BCC_F = \beta^*$. Now, $\angle \overline{C}CD = \gamma - 2\beta^* = (\pi - (\alpha + \beta)) - 2(\pi/2 - \beta) = \beta - \alpha$. ■

Corollary 10. *The length of the chord from a vertex along the altitude has length* $C\overline{C} = \cos(\alpha - \beta)$.

4. DILATION. Two cases of dilation by factors of 2, centered at H, are examined here.

Proposition 11. *Let $\triangle ABC$ be inscribed in a unit circle centered at O. Consider the dilation δ with ratio 2:1 centered at H.*

(i) *δ carries the orthic triangle $\triangle A_F B_F C_F$ onto the circum-orthic triangle, whose vertices are $\overline{A}$, $\overline{B}$, and $\overline{C}$.*

(ii) *δ^{-1} carries the unit circle into the circle circumscribing the orthic triangle. Its radius is $1/2$ and its center is at the midpoint N of the segment OH.*

Proof. (i) By Proposition 4, $|HA_F| = |A_F\overline{A}|$ (see Figure 7), and similarly for the other two altitudes, hence (i) follows. (ii) $\triangle\overline{A}\,\overline{B}\,\overline{C}$ is inscribed in the circle of diameter 1 and center O, so the inverse dilation carries O to N and the circumcircle of $\triangle\overline{A}\,\overline{B}\,\overline{C}$ to the circumcircle of $\triangle A_F B_F C_F$ with radius $1/2$ (see Figure 12). ∎

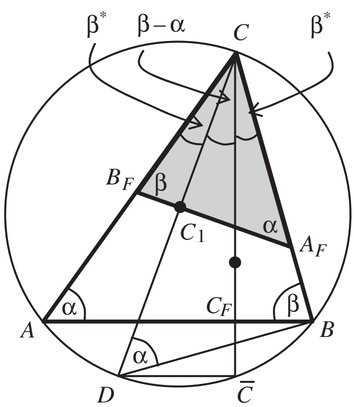


Figure 11. Angle differences.

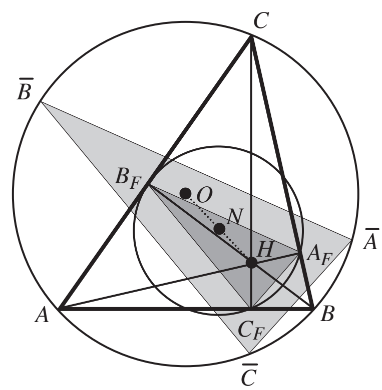


Figure 12. Orthic and circum-orthic triangle. Unit circle and 9-point circle.

Among the circles associated with a triangle one is exceptionally famous: the circumcircle of the orthic triangle. It is called the **nine-point circle**, as it also contains the midpoints of the sides and the Euler points. Three different proofs of this property, appearing in the first pages of Chapter I of [**6**], stress the elementary nature of the proposition. It also surfaces here as a natural consequence of our trigonometric considerations.

Proposition 12 (Nine-Point Circle Theorem). *The circle centered at N with radius $1/2$ contains all midpoints of sides of $\triangle ABC$, feet of its altitudes, and midpoints of ears of the altitudes.*

Proof. Assume that $\alpha \neq \beta$. Reduce the triangle $C\overline{C}D$ by a factor of 2 by contraction δ^{-1} centered at H (see Figure 13). The vertices are mapped as follows: $\overline{C} \to C_F$ and $C \to M_{CH}$ (midpoint of the ear). The third vertex, D, maps to the midpoint M_{AB} of AB. To see this, notice that a translation of CO, half the diameter, along the altitude carries it to the hypotenuse of the smaller triangle. Since M_{AB} is the perpendicular projection of O onto AB, it halves the chord. The midpoint of $M_{CH}M_{AB}$ is the intersection point of the diagonals of the parallelogram $HM_{AB}OM_{CH}$. It is also the center of the circumcircle of $\triangle C_F M_{AB} M_{CH}$. Neither its radius $1/2$ nor its center N depend on the choice of the altitude, so the claim holds. ∎

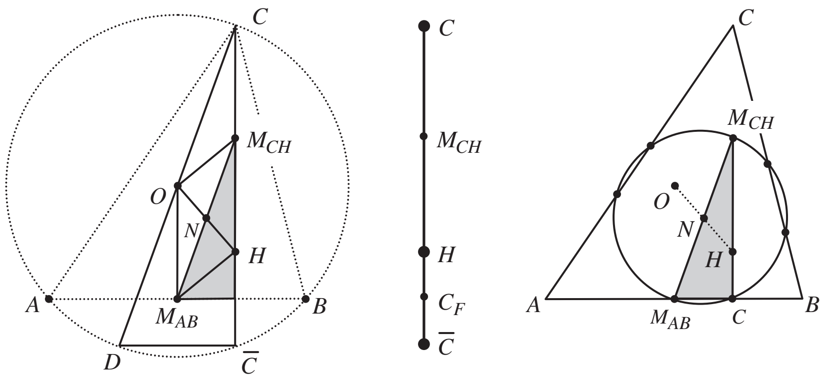


Figure 13. Nine-point circle.

5. IDENTITIES. Putting together the sides and altitudes of $\triangle ABC$ yields trigonometric identities for compound angles:

1. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
2. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
3. $\cos(\alpha - \beta) = \sin\alpha\sin\beta + \cos\alpha\cos\beta$
4. $2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$
5. $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta}$
6. $\cot\alpha + \cot\beta = \dfrac{\sin(\alpha + \beta)}{\sin\alpha\sin\beta}$

Proof. For $\gamma = \pi - (\alpha + \beta)$ we have $\sin\gamma = \sin(\alpha + \beta)$ and $\cos\gamma = -\cos(\alpha + \beta)$, so a look at Figure 14 makes it clear that (1) and (2) hold. Identities (3) and (4) are illustrated in Figure 15. (The lengths of respective segments are given by Proposition 6 and Corollary 7, while $\cos(\alpha - \beta)$ may be read off from Figure 11.)

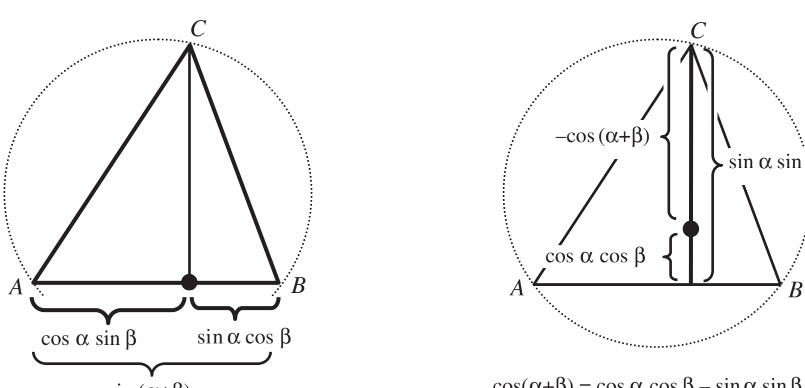
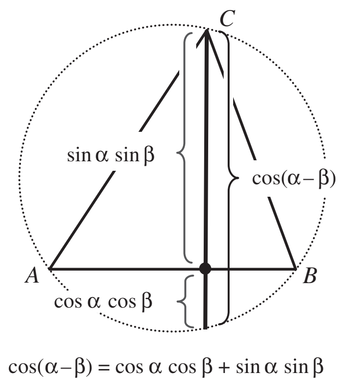


Figure 14. Two identities visualized in the unit circle.

For (5), use the first part of Proposition 6 and Figure 5, and to get (6) use the second part of Proposition 6:



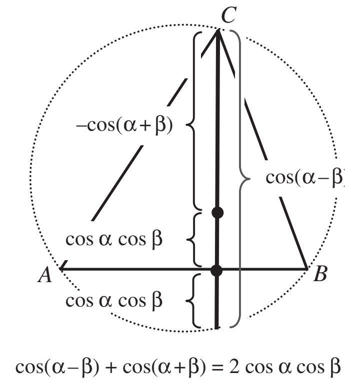


Figure 15. More indentities.

5. $\tan\beta + \tan\alpha = \dfrac{|AC_F|}{|C_F H|} + \dfrac{|C_F B|}{|C_F H|} = \dfrac{|AB|}{|C_F H|} = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.
6. $\cot\alpha + \cot\beta = \dfrac{|AC_F|}{|C_F C|} + \dfrac{|C_F B|}{|C_F C|} = \dfrac{|AB|}{|C_F C|} = \dfrac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$. ■

6. CODA. The three vertices of a triangle and its orthocenter form a quartet of points that catches the essence of triangle geometry: each of these four points is the orthocenter of the triangle formed by the other three. This fact will allow us to unify some of our results, as well as to answer the last pending question.

A **unit orthotetrad** (see [**7**]) is a system of four planar points, P, Q, R, S, such that any three of them form a triangle with unit excircle and any coupling of points into pairs creates two orthogonal segments (see Figure 6). We shall use the following general notation: $\triangle P := \triangle QRS$ for triangles and $\angle QP := \angle RQS$ for angles. Moreover, $[PQ, RS]$ stands for a segment the meaning of which depends on the context: it is the altitude in $\triangle P$ from vertex Q towards side RS, which is equivalent to the stem in $\triangle Q$ from P, and which is also the "pedal segment" QS_F of side PQ in $\triangle R$.

A look at Figure 5 reveals that either $\angle PQ = \angle QP$ or $\angle PQ = \pi - \angle QP$. All characteristic elements (15 angles, 6 sides/ears, and 12 altitudes/stems/pedal segments) follow these two rules:

1. *sides or ears:* $|RS| = \sin\angle PQ = \sin\angle QP$.
2. *other segments:* $[PQ, RS] = (\sin\angle PQ)\cdot(\sin\angle RS)$.

Recall that the results of this paper were all derived for acute triangles. They are naturally transfered to any obtuse triangle when one embeds it into the associated orthotetrad. Note also that any two mutually orthogonal segments in the orthotetrad correspond to the sine and cosine of an angle.

Various other geometric facts easily follow as corollaries from the path advocated in this paper. Below, we present several of them and invite the reader to extend the list. Hints for their proofs may be found at [**4**].

Additional geometrical facts.

1. The area of a triangle is $(ABC) = \frac{1}{2}\sin\alpha\sin\beta\sin\gamma$.
2. $4\sin\alpha\sin\beta\sin\gamma < \pi$.
3. The power of H in the circumcircle of $\triangle ABC$ (product of any two parts of a chord passing through the point) is $2\cos\alpha\cos\beta\cos\gamma$.

4. The distance of the antipodal point of C from AB is equal to the root h_C.
5. The circle inscribed in the orthic triangle has radius $\cos\alpha\cos\beta\cos\gamma$ and center H.
6. The orthocenters of $\triangle AC_FB_F$, $\triangle BA_FC_F$, and $\triangle CB_FA_F$ lie on the sides of the circum-orthic triangle.
7. The distance $u = |OH|$ satisfies $u^2 = 1/4 - 2\cos\alpha\cos\beta\cos\gamma$.
8. For $\alpha+\beta+\gamma=\pi$ one has $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma = 1$.
9. The heights of a triangle obey the inequality
$$\frac{1}{|h_A|}+\frac{1}{|h_B|} > \frac{1}{|h_C|}.$$
10. The inradius r of $\triangle ABC$ satisfies
$$\frac{1}{r} = \frac{1}{|h_A|}+\frac{1}{|h_B|}+\frac{1}{|h_C|}.$$
11. The centers of the circumcircles of $\triangle ABH$, $\triangle BCH$, and $\triangle CAH$ form a triangle $O_{AB}O_{BC}O_{CA}$ congruent to the triangle ABC and they are interchanged by a half-turn around N.

Another interesting exercise is to rewrite the whole story in terms of complex numbers.

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