Chapter 11
More Results

11.1 Hints and Solutions to Selected Problems

2.1. c) The histograms should become more like a normal distribution as n increases from 1 to 200. In particular, when $n = 1$ the histogram should be right skewed while for $n = 200$ the histogram should be nearly symmetric. Also the scale on the horizontal axis should decrease as n increases.

d) Now $\overline{Y} \sim N(0, 1/n)$. Hence the histograms should all be roughly symmetric, but the scale on the horizontal axis should be from about $-3/\sqrt{n}$ to $3/\sqrt{n}$.

2.3. a) $E(X) = \frac{3\theta}{\theta+1}$, thus
$\sqrt{n}(\overline{X} - E(X)) \xrightarrow{D} N(0, V(X))$, where
$V(X) = \frac{9\theta}{(\theta+2)(\theta+1)^2}$. Let $g(y) = \frac{y}{3-y}$, thus $g'(y) = \frac{3}{(3-y)^2}$. Using the delta method, $\sqrt{n}(T_n - \theta) \xrightarrow{D} N(0, \frac{\theta(\theta+1)^2}{\theta+2})$.

b) It is asymptotically efficient if $\sqrt{n}(T_n - \theta) \xrightarrow{D} N(0, \nu(\theta))$, where

$$\nu(\theta) = \frac{\frac{d}{d\theta}(\theta)}{-E(\frac{d^2}{d\theta^2} lnf(x|\theta))}.$$

But, $E((\frac{d^2}{d\theta^2} lnf(x|\theta)) = \frac{1}{\theta^2}$. Thus $\nu(\theta) = \theta^2 \neq \frac{\theta(\theta+1)^2}{\theta+2}$.

c) $\overline{X} \to \frac{3\theta}{\theta+1}$ in probability. Thus $T_n \to \theta$ in probability.

2.5. See Example 2.9.

2.7. a) See Example 2.8.

2.12. a) $Y_n \stackrel{D}{=} \sum_{i=1}^n X_i$ where the X_i are iid χ^2_1. Hence $E(X_i) = 1$ and $\text{Var}(X_i) = 2$. Thus by the CLT,

$$\sqrt{n}\left(\frac{Y_n}{n}-1\right) \stackrel{D}{=} \sqrt{n}\left(\frac{\sum_{i=1}^n X_i}{n}-1\right) \xrightarrow{D} N(0,2).$$

b) Let $g(\theta)=\theta^3$. Then $g'(\theta)=3\theta^2$, $g'(1)=3$, and by the delta method,

$$\sqrt{n}\left[\left(\frac{Y_n}{n}\right)^3-1\right] \xrightarrow{D} N(0,2(g'(1))^2)=N(0,18).$$

2.22. See the proof of Theorem 1.33.

2.26. a) See Example 2.1b.
b) See Example 2.3.
c) See Example 2.6.

2.27. a) By the CLT, $\sqrt{n}(\overline{X}-\lambda)/\sqrt{\lambda} \xrightarrow{D} N(0,1)$. Hence $\sqrt{n}(\overline{X}-\lambda) \xrightarrow{D} N(0,\lambda)$.

b) Let $g(\lambda)=\lambda^3$ so that $g'(\lambda)=3\lambda^2$ then $\sqrt{n}[(\overline{X})^3-(\lambda)^3] \xrightarrow{D} N(0,\lambda[g'(\lambda)]^2)= N(0,9\lambda^5)$.

2.28. a) $\overline{X}$ is a complete sufficient statistic. Also, we have $\frac{(n-1)S^2}{\sigma^2}$ has a chi square distribution with $df=n-1$, thus since σ^2 is known the distribution of S^2 does not depend on μ, so S^2 is ancillary. Thus, by Basu's Theorem $\overline{X}$ and S^2 are independent.

b) by CLT (n is large) $\sqrt{n}(\overline{X}-\mu)$ has approximately normal distribution with mean 0 and variance σ^2. Let $g(x)=x^3$, thus, $g^{'}(x)=3x^2$. Using delta method $\sqrt{n}(g(\overline{X})-g(\mu))$ goes in distribution to $N(0,\sigma^2(g^{'}(\mu))^2)$ or $\sqrt{n}(\overline{X}^3-\mu^3)$ goes in distribution to $N(0,\sigma^2(3\mu^2)^2)$.

2.29. a) According to the standard theorem, $\sqrt{n}(\hat{\theta}_n-\theta)\to N(0,3)$.

b) $E(Y)=\theta, Var(Y)=\frac{\pi^2}{3}$, according to CLT we have $\sqrt{n}(\overline{Y}_n-\theta)\to N(0,\frac{\pi^2}{3})$.

c) $MED(Y)=\theta$, then $\sqrt{n}(MED(n)-\theta)\to N(0,\frac{1}{4f^2(MED(Y))})$ and $f(MED(Y))=\frac{\exp(-(\theta-\theta))}{[1+\exp(-(\theta-\theta))]^2}=\frac{1}{4}$. Thus $\sqrt{n}(MED(n)-\theta)\to N(0,\frac{1}{4\frac{1}{16}})\to \sqrt{n}(MED(n)-\theta)\to N(0,4)$.

d) All three estimators are consistent, but $3<\frac{\pi^2}{3}<4$, therefore the estimator $\hat{\theta}_n$ is the best, and the estimator MED(n) is the worst.

2.31. a) $F_n(y)=0.5+0.5y/n$ for $-n<y<n$, so $F(y)\equiv 0.5$.
b) No, since $F(y)$ is not a cdf.

2.32. a) $F_n(y)=y/n$ for $0<y<n$, so $F(y)\equiv 0$.
b) No, since $F(y)$ is not a cdf.

2.33. a)

$$\sqrt{n}\ \left(\ \overline{Y} - \ \frac{1-\rho}{\rho}\ \right) \xrightarrow{D} N\left(0, \frac{1-\rho}{\rho^2}\right)$$

by the CLT.

c) The method of moments estimator of ρ is $\hat{\rho} = \frac{1}{1+\overline{Y}}$.

d) Let $g(\theta) = 1 + \theta$ so $g'(\theta) = 1$. Then by the delta method,

$$\sqrt{n}\ \left(\ g(\overline{Y}) - \ g(\frac{1-\rho}{\rho})\ \right) \xrightarrow{D} N\left(0, \frac{1-\rho}{\rho^2}1^2\right)$$

or

$$\sqrt{n}\ \left(\ (1+\overline{Y}) - \ \frac{1}{\rho}\ \right) \xrightarrow{D} N\left(0, \frac{1-\rho}{\rho^2}\right).$$

This result could also be found with algebra since $1+\overline{Y} - \frac{1}{\rho} = \overline{Y} + 1 - \frac{1}{\rho} = \overline{Y} + \frac{\rho-1}{\rho} = \overline{Y} - \frac{1-\rho}{\rho}$.

e) $\overline{Y}$ is the method of moments estimator of $E(Y) = (1-\rho)/\rho$, so $1+\overline{Y}$ is the method of moments estimator of $1 + E(Y) = 1/\rho$.

2.34. a) $\sqrt{n}(\bar{X} - \mu)$ is approximately $N(0, \sigma^2)$. Define $g(x) = \frac{1}{x}$, $g'(x) = \frac{-1}{x^2}$. Using delta method, $\sqrt{n}(\frac{1}{\bar{X}} - \frac{1}{\mu})$ is approximately $N(0, \frac{\sigma^2}{\mu^4})$. Thus $1/\overline{X}$ is approximately $N(\frac{1}{\mu}, \frac{\sigma^2}{n\mu^4})$, provided $\mu \neq 0$.

b) Using part a)

$\frac{1}{\overline{X}}$ is asymptotically efficient for $\frac{1}{\mu}$ if

$$\frac{\sigma^2}{\mu^4} = \left[\frac{\big(\tau'(\mu)\big)^2}{E_\mu\left(\frac{\partial}{\partial\mu}\ln f(X/\mu)\right)^2}\right]$$

$$\tau(\mu) = \frac{1}{\mu}$$

$$\tau'(\mu) = \frac{-1}{\mu^2}$$

$$\ln f(x|\mu) = \frac{-1}{2}\ln 2\pi\sigma^2 - \frac{(x-\mu)^2}{2\sigma^2}$$

$$E\left[\frac{\partial}{\partial\mu}\ln f(X/\mu)\right]^2 = \frac{E(X-\mu)^2}{\sigma^4} = \frac{1}{\sigma^2}$$

Thus

$$\frac{\left(\tau'(\mu)\right)^2}{E_\mu\left[\frac{\partial}{\partial\mu}\ln f(X/\mu)\right]^2} = \frac{\sigma^2}{\mu^4}.$$

2.35. a) $E(Y^k) = 2\theta^k/(k+2)$ so $E(Y) = 2\theta/3$, $E(Y^2) = \theta^2/2$ and $V(Y) = \theta^2/18$. So $\sqrt{n}\ \left(\ \overline{Y} - \frac{2\theta}{3}\ \right) \xrightarrow{D} N\left(0, \frac{\theta^2}{18}\right)$ by the CLT.

b) Let $g(\tau) = \log(\tau)$ so $[g'(\tau)]^2 = 1/\tau^2$ where $\tau = 2\theta/3$. Then by the delta method,
$\sqrt{n}\ \left(\ \log(\overline{Y}) - \log\left(\frac{2\theta}{3}\right)\ \right) \xrightarrow{D} N\left(0, \frac{1}{8}\right)$.

c) $\hat{\theta}^k = \frac{k+2}{2n}\sum Y_i^k$.

2.36. a) $\sqrt{n}\ \left(\ \overline{Y} - \ \frac{r(1-\rho)}{\rho}\ \right) \xrightarrow{D} N\left(0, \frac{r(1-\rho)}{\rho^2}\right)$ by the CLT.

b) Let $\theta = r(1-\rho)/\rho$. Then

$$g(\theta) = \frac{r}{r + \frac{r(1-\rho)}{\rho}} = \frac{r\rho}{r\rho + r(1-\rho)} = \rho = c.$$

Now

$$g'(\theta) = \frac{-r}{(r+\theta)^2} = \frac{-r}{(r + \frac{r(1-\rho)}{\rho})^2} = \frac{-r\rho^2}{r^2}.$$

So

$$[g'(\theta)]^2 = \frac{r^2\rho^4}{r^4} = \frac{\rho^4}{r^2}.$$

Hence by the delta method

$$\sqrt{n}\ (\ g(\overline{Y}) - \rho\ ) \xrightarrow{D} N\left(0, \frac{r(1-\rho)}{\rho^2}\frac{\rho^4}{r^2}\right) = N\left(0, \frac{\rho^2(1-\rho)}{r}\right).$$

c) $\overline{Y} \stackrel{\text{set}}{=} r(1-\rho)/\rho$ or $\rho\overline{Y} = r - r\rho$ or $\rho\overline{Y} + r\rho = r$ or $\hat{\rho} = r/(r+\overline{Y})$.

2.37. a) By the CLT,

$$\sqrt{n}\ \left(\ \overline{X} - \frac{\theta}{2}\ \right) \xrightarrow{D} N\left(0, \frac{\theta^2}{12}\right).$$

b) Let $g(y) = y^2$. Then $g'(y) = 2y$ and by the delta method,

$$\sqrt{n}\ \left(\overline{X}^2 - (\frac{\theta}{2})^2\ \right) = \sqrt{n}\ \left(\overline{X}^2 - \frac{\theta^2}{4}\ \right) = \sqrt{n}\ \left(\ g(\overline{X}) - g(\frac{\theta}{2})\ \right) \xrightarrow{D}$$

$$N\left(0, \frac{\theta^2}{12}\ [g'(\frac{\theta}{2})]^2\right) = N\left(0, \frac{\theta^2}{12}\ \frac{4\theta^2}{4}\right) = N\left(0, \frac{\theta^4}{12}\right).$$

2.38. a) $E(X_i) = \beta/(\beta + \beta) = 1/2$ and $V(X_i) = \dfrac{\beta^2}{(2\beta)^2(2\beta+1)} = \dfrac{1}{4(2\beta+1)} = \dfrac{1}{8\beta+4}$. So

$$\sqrt{n}\left(\overline{X}_n - \frac{1}{2} \right) \xrightarrow{D} N\left(0, \frac{1}{8\beta+4}\right)$$

by the CLT.

b) Let $g(x) = \log(x)$. So $d = g(1/2) = \log(1/2)$. Now $g'(x) = 1/x$ and $(g'(x))^2 = 1/x^2$. So $(g'(1/2))^2 = 4$. So

$$\sqrt{n}(\ \log(\overline{X}_n) - \log(1/2)\ ) \xrightarrow{D} N\left(0, \frac{1}{8\beta+4}\ 4\right) = N\left(0, \frac{1}{2\beta+1}\right)$$

by the delta method.

11.2 Tables

Tabled values are F(0.95,k,d) where $P(F < F(0.95, k, d)) = 0.95$.
00 stands for ∞. Entries produced with the `qf(.95,k,d)` command in *R*. The numerator degrees of freedom are k while the denominator degrees of freedom are d.

k d	1	2	3	4	5	6	7	8	9	00
1	161	200	216	225	230	234	237	239	241	254
2	18.5	19.0	19.2	19.3	19.3	19.3	19.4	19.4	19.4	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.41
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	1.84
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	1.62
00	3.84	3.00	2.61	2.37	2.21	2.10	2.01	1.94	1.88	1.00

Tabled values are $t_{\alpha,d}$ where $P(t < t_{\alpha,d}) = \alpha$ where t has a t distribution with d degrees of freedom. If $d > 29$ use the $N(0,1)$ cutoffs $d = Z = \infty$.

					alpha					pvalue
d	0.005	0.01	0.025	0.05	0.5	0.95	0.975	0.99	0.995	left tail
1	-63.66	-31.82	-12.71	-6.314	0	6.314	12.71	31.82	63.66	
2	-9.925	-6.965	-4.303	-2.920	0	2.920	4.303	6.965	9.925	
3	-5.841	-4.541	-3.182	-2.353	0	2.353	3.182	4.541	5.841	
4	-4.604	-3.747	-2.776	-2.132	0	2.132	2.776	3.747	4.604	
5	-4.032	-3.365	-2.571	-2.015	0	2.015	2.571	3.365	4.032	
6	-3.707	-3.143	-2.447	-1.943	0	1.943	2.447	3.143	3.707	
7	-3.499	-2.998	-2.365	-1.895	0	1.895	2.365	2.998	3.499	
8	-3.355	-2.896	-2.306	-1.860	0	1.860	2.306	2.896	3.355	
9	-3.250	-2.821	-2.262	-1.833	0	1.833	2.262	2.821	3.250	
10	-3.169	-2.764	-2.228	-1.812	0	1.812	2.228	2.764	3.169	
11	-3.106	-2.718	-2.201	-1.796	0	1.796	2.201	2.718	3.106	
12	-3.055	-2.681	-2.179	-1.782	0	1.782	2.179	2.681	3.055	
13	-3.012	-2.650	-2.160	-1.771	0	1.771	2.160	2.650	3.012	
14	-2.977	-2.624	-2.145	-1.761	0	1.761	2.145	2.624	2.977	
15	-2.947	-2.602	-2.131	-1.753	0	1.753	2.131	2.602	2.947	
16	-2.921	-2.583	-2.120	-1.746	0	1.746	2.120	2.583	2.921	
17	-2.898	-2.567	-2.110	-1.740	0	1.740	2.110	2.567	2.898	
18	-2.878	-2.552	-2.101	-1.734	0	1.734	2.101	2.552	2.878	
19	-2.861	-2.539	-2.093	-1.729	0	1.729	2.093	2.539	2.861	
20	-2.845	-2.528	-2.086	-1.725	0	1.725	2.086	2.528	2.845	
21	-2.831	-2.518	-2.080	-1.721	0	1.721	2.080	2.518	2.831	
22	-2.819	-2.508	-2.074	-1.717	0	1.717	2.074	2.508	2.819	
23	-2.807	-2.500	-2.069	-1.714	0	1.714	2.069	2.500	2.807	
24	-2.797	-2.492	-2.064	-1.711	0	1.711	2.064	2.492	2.797	
25	-2.787	-2.485	-2.060	-1.708	0	1.708	2.060	2.485	2.787	
26	-2.779	-2.479	-2.056	-1.706	0	1.706	2.056	2.479	2.779	
27	-2.771	-2.473	-2.052	-1.703	0	1.703	2.052	2.473	2.771	
28	-2.763	-2.467	-2.048	-1.701	0	1.701	2.048	2.467	2.763	
29	-2.756	-2.462	-2.045	-1.699	0	1.699	2.045	2.462	2.756	
Z	-2.576	-2.326	-1.960	-1.645	0	1.645	1.960	2.326	2.576	
CI						90%	95%		99%	
	0.995	0.99	0.975	0.95	0.5	0.05	0.025	0.01	0.005	right tail
	0.01	0.02	0.05	0.10	1	0.10	0.05	0.02	0.01	two tail