Exam 2 on Wednesday, March 9 covers sections 6.1, 6.2, 6.3, 6.4, 6.5, 7.1, 7.2, 7.3, 7.4 and 7.5. You are allowed a NON-GRAPHICAL calculator but NO NOTES. The common final problems from chapter 6 may again appear on exam 2:

1) **Common Final Problem:** You are given a story problem with 3 sets A, B, and C. From the story problem, from *top to bottom list the counts* of the biggest set U, then of the 3 single sets, then of the 3 pairwise intersections, then the 3 way intersection: $c(U), c(A), c(B), c(C), c(A \cap B), c(A \cap C), c(B \cap C)$ and $c(A \cap B \cap C)$. Then you will draw a Venn diagram or be given a Venn diagram of the three sets. This Venn diagram divides U into 8 regions of the form $A^* \cap B^* \cap C^*$ where $A^* = A$ if the region is contained in set A and $A^* = \overline{A}$ otherwise. Next *from the bottom* of the counts *work up*, filling in the counts of the eight regions. S04 1, F04, 3, F03 5, F02 12, S02 7, F01 5, Q1, Q2, E1 1.

2) **Common Final Problem:** Use the multiplication principle to find *how many* ways to perform the tasks. Important special cases include finding the number of license plates or serial numbers. See S04 6, F03 6, S03 4,7, Q2 4,5, Q3 7, HW2, HW3, E1 6,7,9.

3) **Common Final Problem.** A story problem says "how many ways can r *different* objects be *distributed* to n (people or) children in such a way *that no child gets more than one object*?" Solution: $P(n,r)$ ways. See S03 7. Q3 6, HW 3 41 E1 12.

4) **Common Final Problem.** Compute $C(n,r) = \binom{n}{r}, P(n,r)$, and $n!$ showing all work. F04 1, S04 2, F03 1, F02 2, S02 2, F01 1, Q3 4, HW3 2,3,7, HW4 37, 45, 52, 55, E1 4.

5) **Common Final Problem.** Often the multiplication principle will be combined with combinations, permutations, powers and factorials. F04 6, F02 10, S02 15, F01 6, HW4 80, HW3 40a, HW4 38, E1 6.

6) **Common Final Problem.** How many n-letter words (real or imaginary) can be formed from an n-letter word such as MISSISSIPPI? Each letter is a kind: 1 M, 4 I's, 4 S's, 2 P's. There are 11 letters total, so $\frac{11!}{1!4!4!2!}$. S04 3, S03 6, Q3 5, HW4 23,24, E1 8.

7) **Common Final Problem.** Story problem has k groups of indistinguishable objects of size n_i, $i = 1, ..., k$. *Color* is often used, eg green, red and white balls, lights, or flags. Find the total number of objects $n = n_1 + ... + n_k$. Then the number of ways to arrange the objects is $\frac{n!}{n_1!n_2! \cdots n_k!}$. S03 9, HW4 25,26.

8) **Common Final Problem.** A coin is tossed n times. How many outcomes are possible? Solution: 2^n.

9) **Common Final Problem.** A hand of k cards is dealt from a deck of 52 cards. How many k-card hands are there? Solution: $C(52,k)$. See S04 4a.

SEE REVIEW FOR EXAM 1 FOR MORE INFORMATION.

Review for chapter 7:

The *sample space* S is the set of all possible outcomes of an experiment (and takes the place of the universal set in set theory). One way to find a sample space is to use order to list all possibilities in S. A tree diagram can also help. Tossing 2 die, and a coin two or three times are typical examples. An *event* is any subset of $S = \{e_1, ..., e_n\}$. An event with exactly one element is a simple event. The *frequency interpretation of probability* states that the *probability* of outcome e_k is the proportion of times that e_k would occur if the experiment was performed again and again (infinitely often).

Know: If E is an event, then $0 \leq P(E) \leq 1$.

The *empty set* $\emptyset$ is the set that contains no elements, and $P(\emptyset) = 0$. If $P(E) = 0$, then event E is impossible. If $P(E) = 1$, then event E is certain to occur. $P(S) = 1$. In general, $P(E)$ is the sum of the probabilities of the simple events in E. So if $E = \{e_1, e_2, ..., e_k\}$, then $P(E) = P(e_1) + P(e_2) + \cdots + P(e_k)$.

10) **Common Final Problem.** Given a story problem, list the outcomes that make up an event (especially for die problems). S04 8, F03 7 (die table), HW5 4, 7, 39, 16 and the technique is useful for many other problems.

Know. Events E and F are **mutually exclusive** if E and F have no outcomes in common: $E \cap F = \emptyset$. Events are mutually exclusive if they are disjoint sets.

Additive rule for mutually exclusive events: If E and F are mutually exclusive events, then $P(E \cup F) = P(E) + P(F)$ since $P(E \cap F) = P(\emptyset) = 0$. If $E_1, E_2, ..., E_k$ are mutually exclusive events, then $P(E_1 \cup E_2 \cup \cdots \cup E_k) = P(E_1) + P(E_2) + \cdots + P(E_k)$.

If $S = \{e_1, e_2, ..., e_n\}$, then $\{e_1\}, \{e_2\}, ..., \{e_n\}$ are mutually exclusive. If the occurrence of event E prevents the occurrence of event F and vice verca, then E and F are mutually exclusive.

Complement rule: $P(\overline{E}) = 1 - P(E)$. Hence $P(E) = 1 - P(\overline{E})$. This rule is useful because often one probability is far easier to find directly than the other.

Let $S = \{e_1, ..., e_n\}$. Then the outcomes $e_1, ..., e_n$ are *equally likely* if $P(e_i) = 1/n$ for each $i = 1, ..., n$, and if event E contains m outcomes, then $P(E) = \frac{m}{n} = \frac{c(E)}{c(S)}$. For this reason, counting techniques of chapter 6 are still needed for chapter 7. Equally likely outcomes often occur when items are selected *randomly* (eg cards), and when *fair* coins and die are tossed.

11) **Common Final Problem.** If a *fair coin* is tossed n times, then the probability of (exactly) r heads (or of r tails) is $\frac{C(n,r)}{2^n}$. S03 5a, F02 5a Q5 3, HW6 3, 4a.

12) **Common Final Problem.** If a *fair coin* is tossed n times, find the *probability of at least j heads* or the *probability of at most j tails.* Let P(k) be the probability of k heads (or tails) in n tosses. Then P(**at least j** H's) = P(between j and n heads) $= P(j) + P(j+1) + \cdots + P(n) = 1 - P(0) - P(1) - \cdots - P(j-1)$. In particular, P(**at least one** H) $= 1 - P(0) = 1 - 1/2^n$.

Similarly, P(**at most j heads**) = P(between 0 and j heads) $= P(0) + P(1) + \cdots + P(j) = 1 - P(j+1) - P(j+2) - \cdots - P(n)$. Notice that $P(j) = \frac{C(n,j)}{2^n}$. That is, use common final problem (11) repeatedly. S03 5b, F02 5b, HW6 6.

13) **Common Final Problem.** There are two groups with n_1 and n_2 members where $n_1 + n_2 = n$. A committee or subgroup of size r is formed by selecting r_1 members from the 1st group and r_2 members from the 2nd group where $r_1 + r_2 = r$. The probability of obtaining such a group if the r committee members were randomly selected from the n is equal to

$$\frac{C(n_1, r_1)C(n_2, r_2)}{C(n, r)}.$$

Defectives and nondefectives, females and males, yellow balls and non-yellow balls, spades and non-spades, queens and non-queens, etc are common ways of getting two groups. See

F04 2ab, S04 4bc, F03 4a, F02 4a, S02 16a F01 4a, Q5 2a, HW6 1, 2, 24, HW8 13ab.

14) **Common Final Problem.** Again there are two groups as in common final problem 13), say n_1 males and n_2 females. Again r people are selected from both groups. Let P(k) be the probability that there are k M's and $r-k$ F's in the group. Then P(**at least j males**) $= P(j) + P(j+1) + \cdots + P(r) =$

$$\frac{C(n_1,j)C(n_2,r-j)}{C(n,r)} + \frac{C(n_1,j+1)C(n_2,r-j-1)}{C(n,r)} + \cdots + \frac{C(n_1,r)C(n_2,0)}{C(n,r)} =$$

$1 - P(0) - P(1) - \cdots - P(j-1) =$

$$1 - \frac{C(n_1,0)C(n_2,r)}{C(n,r)} - \frac{C(n_1,1)C(n_2,r-1)}{C(n,r)} - \cdots - \frac{C(n_1,j-1)C(n_2,r-j+1)}{C(n,r)}.$$

That is, use common final problem 13) repeatedly with $r_1 = j, j+1, ..., r$ and $r_2 = r - r_1$.
In particular, P(**at least one** M) $= 1 - P(0) = 1 - \dfrac{C(n_1,0)C(n_2,r)}{C(n,r)}$.
Similarly P(**at most j males**) $= P(0) + P(1) + \cdots + P(j) =$

$$\frac{C(n_1,0)C(n_2,r)}{C(n,r)} + \frac{C(n_1,1)C(n_2,r-1)}{C(n,r)} + \cdots + \frac{C(n_1,j)C(n_2,r-j)}{C(n,r)} =$$

$1 - P(j+1) - P(j+2) - \cdots - P(r) =$

$$1 - \frac{C(n_1,j+1)C(n_2,r-j-1)}{C(n,r)} - \frac{C(n_1,j+2)C(n_2,r-j-2)}{C(n,r)} - \cdots - \frac{C(n_1,r)C(n_2,0)}{C(n,r)}.$$

That is, use common final problem 13) repeatedly with $r_1 = 0, 1, ..., j$ and $r_2 = r - r_1$.

P(**all members are of the same group** (**eg sex**)) $= \dfrac{\mathrm{C(n_1,r)C(n_2,0)}}{\mathrm{C(n,r)}} + \dfrac{\mathrm{C(n_1,0)C(n_2,r)}}{\mathrm{C(n,r)}}$.

See F04 2c, F03 4bc, S03 8b, F02 4b, S02 16bc, F01 4bc, Q5 1b, HW6 1, HW8 13c.

Know. Let E and F be events from sample space S with $P(F) > 0$. Then the **conditional probability of E given F** is $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$.

Know. Let E and F be events in S. Suppose **any one** of the following conditions holds I1) $P(E \cap F) = P(E)P(F)$, I2) $P(E|F) = P(E)$, or I3) $P(F|E) = P(F)$. Then E and F are **independent** events. If any of the three conditions fails, then E and F are *dependent. Interpretation of independence:* Suppose that $P(E|F) = P(E)$, then knowledge of the occurrence or non-occurrence of F does not change the knowledge of the occurrence or non-occurrence of E.

15) **Common Final Problem.** Determine if events A and B are independent. If $P(A \cap B) = P(A)P(B)$ or if $P(A|B) = P(A)$ (or if $P(B|A) = P(B)$), then A and B are independent. *Only one of these conditions needs to be checked.* If $P(A \cap B) \neq P(A)P(B)$ or if $P(A|B) \neq P(A)$ (or if $P(B|A) \neq P(B)$), then A and B are dependent. Again, only one of these conditions needs to be checked. If $P(E) > 0$, $P(F) > 0$ and if E and F are

mutually exclusive, $P(E|F) = P(E \cap F) = 0$ and E and F are dependent. F04 5d, S04 9d, F02 6c, HW7 5,6, HW8 12

Product rule: $P(E \cap F) = P(F)P(E|F)$.

A **tree diagram** can be useful for finding $P(A_i \cap B_j)$ when there are several A_i and several B_j. **Multiplication rule for independent events:** if E and F are *independent*, then $P(E \cap F) = P(E)P(F)$. If A_1, A_2, ..., A_n are n independent events, then $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2) \cdots P(A_n)$.

General Additive Rule. $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

16) **Common Final Problem.** Given any three of the above probabilities, use the general additive rule to find the fourth probability. Variants: a) given $P(E), P(F)$ and that E and F are **independent**, then $P(E \cup F) = P(E) + P(F) - P(E)P(F)$ while $P(E \cap F) = P(E)P(F)$. b) Given $P(E), P(F)$ and that E and F are **mutually exclusive**, then $P(E \cup F) = P(E) + P(F)$ while $P(E \cap F) = 0$.
c) Given any 2 of $P(E \cap F), P(F), P(E|F)$, use the product rule to find the third.
F02 8a, S03 2, F02 6ab, S02 5, Q3 1a, 2c, Q6, HW5 3, 5, 6, 31, HW8 12

17) **Common Final Problem.** Two events E and F correspond to a **Venn diagram** with 2 circles and 4 regions. Fill out the Venn diagram and then use the results to find various probabilities. The probabilities in 4 regions are $P(E \cap \overline{F}), P(E \cap F), P(\overline{E} \cap F)$ and $P(\overline{E} \cap \overline{F})$. The four regions are mutually exclusive. S04 9, F03 8, F02 6, F01 7b, Q6

18) **Common Final Problem.** List equally likely outcomes and find $P(E) = c(E)/c(S)$ or $P(E|F) = [c(E \cap F)/C(S)]/[c(F)/c(S)] = c(E \cap F)/c(F)$. Often counts are given in a table or you make a table of 3 coin tosses or 2 die tosses.
Toss two die (eg red or green) (or toss a die twice with a 1st die, 2nd die). a) Find the probability that the sum of the two die = k. Solution: fill a table with 36 entries and find the number of entries where the sum is equal to k. These entries lie on a diagonal. Let E_k = "sum of the dice is k". Then $P(E_k)$ = P(sum of the dice is equal to k) $= \dfrac{c(E_k)}{c(S)} = \dfrac{c(E_k)}{36}$ where $c(E_k)$ is the number of outcomes where the two die sum to k.

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k                       |2    3    4    5    6    7    8    9    10   11   12
P(sum of two dice = k)|1/36 2/36 3/36 4/26 5/36 6/36 5/36 4/36 3/36 2/36 1/36
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b) Find $P(E|F)$ where typical E's and F's are "sum of two die = k" "at least one die is a k", "exactly one die is a k", "red die is a k." To solve this problem, list the outcomes in F in the table that has rows 1 to 6 and columns 1 to 6, then from the outcomes in F, find the outcomes in E (these are the outcomes in both E and F). Then the solution is the (number of items in both E and F)/(number of items in F). That is,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{c(E \cap F)/36}{c(F)/36} = \frac{c(E \cap F)}{c(F)}.$$

See F04 4,8,9a, F03 7, S03 1, S02 3, F01 7, Q4 3, Q5 4, Q6, HW5 16, HW7 32.

Key words for conditional probability: **given,** *it is known that*

Key words for chapter 6: *how many*

Key words for ch. 7: *probability*