Exam 4, Wednesday, May 4. The Final is 10:10-12:10, Monday, May 9, Pulliam Hall 42 (center, rows H-O).

The final covers sections 6.1, 6.2, 6.3, 6.4, 6.5, 7.1, 7.2, 7.3, 7.4, 7.5, 8.1, 1.1, 1.2, 1.3, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 3.1, 3.2, 3.3, 4.1, and 4.2. You are allowed a NON-GRAPHICAL calculator but NO NOTES. Turn your cell phone OFF.

* means very likely on FINAL ** means on EXAM 4 AND FINAL

Problems since Exam 3:

1**) **Common Final problem:** a) Find the inverse A^{-1} of a 3×3 matrix A. To find A^{-1}, set up the augmented matrix $\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right] = [A|I_3]$

and put into reduced row-echelon form $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & c_{11} & c_{12} & c_{13} \\ 0 & 1 & 0 & c_{21} & c_{22} & c_{23} \\ 0 & 0 & 1 & c_{31} & c_{32} & c_{33} \end{array}\right] = [I_3|A^{-1}]$.

That is, $A^{-1} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$. See Q11 1, E4, HW18 17, HW22 37, S02 10, F01 11

2**) **Common Final Problem**: Show that A^{-1} is the inverse matrix by showing either that $AA^{-1} = I_3$ or by showing that $A^{-1}A = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. See Q10 1c, Q11 3, F04 12b

3) Common Final Problem: Use the matrix A^{-1} to solve the system of equations

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

or $AX = B$. The solution is $X = A^{-1}B$ where

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, A^{-1}B = \begin{bmatrix} c_{11}b_1 + c_{12}b_12 + c_{13}b_3 \\ c_{21}b_1 + c_{22}b_2 + c_{23}b_3 \\ c_{31}b_1 + c_{32}b_2 + c_{33}b_3 \end{bmatrix}$. See F04 12c, HW 22 17, 37.

Problems from chapter 3.

4**) **Common Final Problem:** Set up a linear programming model from a story problem. See Q12 2, F04 13, S04 15,F03 12, S03 15, F02 7, S02 9, F01 10, HW 21 3.

5**) **Common Final Problem:** Given a linear programming model with two variables, solve geometrically. The solution is at a corner point, one of which is usually hard to find. If (x,y) is the corner point that is hard to find, usually x = y or x and y are both integers. The feasible region has the corner points. To draw the lines for the graph, you need the y- and x- intecepts for each constraint below the objective function.

max $P = 2x + 3y$	objective function	
.	graph	xint
if $x + 2y \leq 10$	$y = \frac{-1}{2}x + 5$	(10,0)
$2x + y \geq 10$	$y = -2x + 10$	(5,0)
$x \geq 0,\ y \geq 0$		

Notice that if a constraint gives the line $Ax + By = C$, then the x-intercept is found by setting y = 0 and the y-intercept is found by setting x = 0, that is, the corresponding line has x-intercept = $(\frac{C}{A}, 0)$ and y-intercept = $(0, \frac{C}{B})$. The feasible region is always in the first quadrant so the y-intercept is always positive. On exams, usually the feasible region is a triangle or trapezoid. Often the point (0,0) is NOT a corner point.

Label each corner point on the graph, and evaluate the objective function at each corner point in order to find the solution (the max or min). See Q11 2, F04 15, S04 17, F03 11, S03 16, F02 8, S02 6, F01 9, HW19, HW20 20, 21, 22, 38

6**) **Common Final problem:** Given a simplex tableau, determine if
a) there is no optimal solution
b) the problem is finished, report the final solution
c) a pivot is required, circle the pivot element and perform the pivot. See Q12 3, F02 16, S02 12, F01 12, HW21 1, 3, 5, HW22 2, 13

To do this problem, find the smallest entry in the bottom (objective) row. This entry corresponds to the pivot column.

a) If the smallest entry in the bottom row is negative and none of the pivot column entries are positive, then there is no optimal solution.

b) If the smallest entry in the bottom row is 0 (or positive) then the problem is finished. The variables in the leftmost column are equal to the corresponding values in the rightmost column while all variables that do not appear in the leftmost column are equal to 0.

c) If the smallest entry in the bottom row is negative but at least one of the entries in the pivot column is positive, then a pivot is required. To determine the pivot row, find all of the positive entries in the pivot column. Divide the entry in the rightmost column by the corresponding positive entry in the pivot column. The row with the smallest quotient is the pivot row. The Pivot element is the entry that is in the intersection of the pivot row and the pivot column. To perform the pivot you need to do row operations to make the pivot entry into a 1 and all entries above and below the pivot entry into zeroes. Each row operation must use the pivot row (so to make an entry above or below the pivot element into a zero, multiply the pivot row by a constant and add it to the correspoinding row where a 0 is desired). After the pivot is performed, the basic variables correspond to a column with a 1 and all 0's above and below the 1. The bottom row always has the objective variable (usually P, sometimes z). Generally the final solution can be found after the pivot.

WHEN PERFORMING A PIVOT, KEEP THE RIGHTMOST COLUMN ENTRIES POSITIVE, IF POSSIBLE.

The following are typical

7) Common final problem. Solve a linear programming problem using the Simplex method.
a) Set the "maximize" line to 0.
b) Add a slack variable to each $\leq$ constraint. If there are 3 such constraints, use s1 for the 1st, s2 for the 2nd and s3 for the 3rd constraint.
c) Assume that there are k x's, usually $k = 2$ or 3, and j slack variables $j = 2$ or 3. Write down the "initial tableau" with headers:
basic, $P, x1, x2, ..., xk, s1, .., sp, b$. (sometimes use RHS instead of b).
The slack variables and P go in the "basic column." Perform a pivot as described in 6**). The to find the variables under "basic" after the pivot, in each row find a 1 in a column where all other entries are 0. The variable on the top of this column should be put in the row under "basic".

Especially important final problems from **Chapter 6:**. (See Exam 1 review for more details).

8**) **Common Final Problem.** Compute $C(n, r), P(n, r)$, and $n!$ showing all work.

9*) **Common Final Problem:** You are given a story problem with 3 sets A, B, and C. From the story problem, from *top to bottom list the counts* of the biggest set U, then of the 3 single sets, then of the 3 pairwise intersections, then the 3 way intersection: $c(U), c(A), c(B), c(C), c(A \cap B), c(A \cap C), c(B \cap C)$ and $c(A \cap B \cap C)$. Then you will draw a Venn diagram or be given a Venn diagram of the three sets. This Venn diagram divides U into 8 regions of the form $A^* \cap B^* \cap C^*$ where $A^* = A$ if the region is contained in set A and $A^* = \overline{A}$ otherwise. Next *from the bottom* of the counts *work up*, filling in the counts of the eight regions.

10**) **Common Final Problem:** Use the multiplication principle to find *how many* ways to perform the tasks. Important special cases include finding the number of license plates or serial numbers.

Especially important final problems from **Chapter 7:**. (See Exam 2 review for more details).

11**) **Common Final Problem. Toss two die** (eg red or green) (or toss a die twice with a 1st die, 2nd die). a) Find the probability that the sum of the two die = k. Solution: fill a table with 36 entries and find the number of entries where the sum is equal to k. These entries lie on a diagonal. Let E_k = "sum of the dice is k". Then $P(E_k)$ = P(sum of the dice is equal to k) $= \dfrac{c(E_k)}{c(S)} = \dfrac{c(E_k)}{36}$ where $c(E_k)$ is the number of outcomes where the two die sum to k.

```
k                     |2    3    4    5    6    7    8    9    10   11   12
P(sum of two dice = k)|1/36 2/36 3/36 4/26 5/36 6/36 5/36 4/36 3/36 2/36 1/36
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b) Find $P(E|F)$ where typical E's and F's are "sum of two die = k" "at least one die is a k", "exactly one die is a k", "red die is a k." To solve this problem, list the outcomes in F in the table that has rows 1 to 6 and columns 1 to 6, then from the outcomes in F, find the outcomes in E (these are the outcomes in both E and F). Then the solution is the (number of items in both E and F)/(number of items in F). That is,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{c(E \cap F)/36}{c(F)/36} = \frac{c(E \cap F)}{c(F)}.$$

c) Find $P(E \cap \overline{F})$. LIST ALL OUTCOMES.**Common Final Problem.** Given a story problem, list the outcomes that make up an event (especially for die problems).

12**) **Common Final Problem. General Additive Rule.**
$P(E \cup F) = P(E) + P(F) - P(E \cap F)$. Given any three of the above probabilities, use the general additive rule to find the fourth probability. USING A PROBABILITY VENN DIAGRAM CAN BE USEFUL. The probabilities in 4 regions are $P(E \cap \overline{F}), P(E \cap F), P(\overline{E} \cap F)$ and $P(\overline{E} \cap \overline{F}) = 1 - P(E \cup F)$. The four regions are mutually exclusive. Also find probabilities like $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$

13**) **Common Final Problem.** There are two groups with n_1 and n_2 members where $n_1 + n_2 = n$. A committee or subgroup of size r is formed by selecting r_1 members from the 1st group and r_2 members from the 2nd group where $r_1 + r_2 = r$. The probability of obtaining such a group if the r committee members were randomly selected from the n is equal to

$$\frac{C(n_1, r_1)C(n_2, r_2)}{C(n, r)}.$$

Defectives and nondefectives, females and males, yellow balls and non-yellow balls, spades and non-spades, queens and non-queens, etc are common ways of getting two groups.

14**) **Common Final Problem.** Again there are two groups as in common final problem 13), say n_1 males and n_2 females. Again r people are selected from both groups. Let P(k) be the probability that there are k M's and $r - k$ F's in the group. Then P(**at least j males**) $= P(j) + P(j + 1) + \cdots + P(r) =$

$$\frac{C(n_1, j)C(n_2, r-j)}{C(n, r)} + \frac{C(n_1, j+1)C(n_2, r-j-1)}{C(n, r)} + \cdots + \frac{C(n_1, r)C(n_2, 0)}{C(n, r)} =$$

$1 - P(0) - P(1) - \cdots - P(j - 1) =$

$$1 - \frac{C(n_1, 0)C(n_2, r)}{C(n, r)} - \frac{C(n_1, 1)C(n_2, r-1)}{C(n, r)} - \cdots - \frac{C(n_1, j-1)C(n_2, r-j+1)}{C(n, r)}.$$

That is, use common final problem 13) repeatedly with $r_1 = j, j+1, ..., r$ and $r_2 = r - r_1$.
In particular, P(**at least one** M) $= 1 - P(0) = 1 - \dfrac{C(n_1, 0)C(n_2, r)}{C(n, r)}$.
Similarly P(**at most j males**) $= P(0) + P(1) + \cdots + P(j) =$

$$\frac{C(n_1, 0)C(n_2, r)}{C(n, r)} + \frac{C(n_1, 1)C(n_2, r-1)}{C(n, r)} + \cdots + \frac{C(n_1, j)C(n_2, r-j)}{C(n, r)} =$$

$1 - P(j + 1) - P(j + 2) - \cdots - P(r) =$

$$1 - \frac{C(n_1, j+1)C(n_2, r-j-1)}{C(n, r)} - \frac{C(n_1, j+2)C(n_2, r-j-2)}{C(n, r)} - \cdots - \frac{C(n_1, r)C(n_2, 0)}{C(n, r)}.$$

That is, use common final problem 13) repeatedly with $r_1 = 0, 1, ..., j$ and $r_2 = r - r_1$.
P(**all members are of the same sex)** =

$$\frac{C(n_1, r)}{C(n, r)} + \frac{C(n_2, r)}{C(n, r)}.$$

Material from Ch. 8: See exam 3 review.

Let $A_1, A_2, ..., A_n$ partition S, and let E be an event in S, then
a) $P(E) = P(A_1)P(E|A_1) + P(A_2)P(E|A_2) + \cdots + P(A_n)P(E|A_n)$ and
b) **Bayes' rule:** $P(A_j|E) = \dfrac{P(A_j \cap E)}{P(E)} = \dfrac{P(A_j)P(E|A_j)}{P(E)}$

$$= \frac{P(A_j)P(E|A_j)}{P(A_1)P(E|A_1) + P(A_2)P(E|A_2) + \cdots + P(A_n)P(E|A_n)}.$$

Usually $n = 2$ or $n = 3$. If $n = 2$, $P(E) = P(A)P(E|A) + P(\overline{A})P(E|\overline{A})$ and $P(A|E) = \dfrac{P(A)P(E|A)}{P(A)P(E|A) + P(\overline{A})P(E|\overline{A})}$.

15**) In a **Bayes' rule** story problem, 2 or more unconditional probabilities are given (or easy to find with the complement rule). Several conditional probabilities are also given (or easy to find with the complement rule). **Make a tree diagram** with the events corresponding to the unconditional events labelling the left branches and the events corresponding to the conditional probabilities labelling the right branches. Above the left branches place the unconditional probabilities and above the right branches place the conditional probabilities. You will be asked to find an unconditional right branch probability and to use Bayes' rule to find P(left branch | right branch).

Tips: the hard conditional probability, P(left branch | right branch), usually appears at the end of the story problem. This tells you how to label the left branches and the right branches of the tree. (The easy conditional probabilities, P(right branch | left branch), can also tell you how to label the tree.) The probabilities of the left branch sum to one. Each subtree of right branches has probabilities that sum to one. Occasionally you are asked to find both a P(right branch | left branch) (directly from the tree) and P(left branch | right branch) (using Bayes rule). **See Q7 3, 4 for how to do Bayes problems.**

Problems from chapter 1:

16**) **Common Final problem:** Given two points (x_1, y_1) and (x_2, y_2), find the **slope-intercept form of the line** $y = mx + b$ passing through the two points, find the **y-intercept** and find the **x-intercept**. **Solution:** The slope intercept equation $y = mx + b$ through the points (x_1, y_1) and (x_2, y_2) has slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. After finding the slope m, find b using the equation $y_1 = mx_1 + b$. That is, $b = y_1 - mx_1$. Then the y-intercept is (0,b) and the x-intercept (a,0) is found by solving $0 = mx + b$ for x: $x = a = -b/m$.

Problem from ch 2:

17**) **Common Final problem:** Given augmented matrices, find the general solution (x, y, z) to the problem. Typically three augmented matrices are given. One has a unique solution, one has no solution, and one has infinitely many solutions. The following examples are illustrative.

i) $\begin{bmatrix} 0 & 1 & 0 & | & -5 \\ 0 & 0 & 1 & | & 0 \\ 1 & 0 & 0 & | & -2 \end{bmatrix}$ $\quad \begin{array}{c} y = -5 \\ z = 0 \\ x = -2 \end{array}$

Hence the general solution is $(x, y, z) = (-2, -5, 0)$.

ii) $\begin{bmatrix} 1 & 0 & 2 & | & -2 \\ 0 & 1 & 1 & | & 3 \\ 0 & 0 & 0 & | & 7 \end{bmatrix}$ $\quad 0 = 7, \textit{impossible}$

Hence there is no solution.

iii) $\begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & 4 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$ $\quad \begin{array}{c} x + 2z = 1 \\ y + 4z = -2 \\ 0 = 0 \end{array}$

Hence the general solution is $(x, y, z) = (1-2z, -2-4z, z)$ where z is any real number.

iv) $\begin{bmatrix} 1 & 3 & 2 & | & 1 \\ 0 & 1 & 4 & | & -2 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$ $\quad \begin{array}{cc} x + 3y + 2z = 1 & \textit{or} \;\; x = 1 - 3(-14) - 2(3) = 37 \\ y + 4z = -2 & \textit{or} \;\; y = -2 - 4(3) = -14 \\ z = 3 & 0 = 0 \end{array}$

not likely on final

4) **Common Final Problem.** Often the multiplication principle will be combined with combinations, permutations, powers and factorials.

5) **Common Final Problem.** How many n-letter words (real or imaginary) can be formed from an n-letter word such as MISSISSIPPI? Each letter is a kind: 1 M, 4 I's, 4 S's, 2 P's. There are 11 letters total, so $\frac{11!}{1!4!4!2!}$.

6) **Common Final Problem.** Story problem has k groups of indistinguishable objects of size n_i, $i = 1, ..., k$. *Color* is often used, eg green, red and white balls, lights, or flags. Find the total number of objects $n = n_1 + ... + n_k$. Then the number of ways to arrange the objects is $\frac{n!}{n_1!n_2! \cdots n_k!}$.

7) **Common Final Problem.** A coin is tossed n times. How many outcomes are possible? Solution: 2^n.

8) **Common Final Problem.** A hand of k cards is dealt from a deck of 52 cards. How many k-card hands are there? Solution: $C(n, k)$.

9) **Common Final Problem.** A story problem says "how many ways can r *different* objects be *distributed* to n (people or) children in such a way *that no child gets more than one object*?" Solution: $P(n, r)$ ways.

15) **Common Final Problem.** Determine if events A and B are independent. If $P(A \cap B) = P(A)P(B)$ or if $P(A|B) = P(A)$ (or if $P(B|A) = P(B)$), then A and B are independent. *Only one of these conditions needs to be checked.* If $P(A \cap B) \neq P(A)P(B)$ or if $P(A|B) \neq P(A)$ (or if $P(B|A) \neq P(B)$), then A and B are dependent. Again, only one of these conditions needs to be checked. If $P(E) > 0$, $P(F) > 0$ and if E and F are mutually exclusive, then E and F are dependent (actually an extreme form of dependence: if you know that F occurred then you know that E did not occur. So $P(E|F) = P(E \cap F) = 0$ if E and F are mutually exclusive).

16) **Common Final Problem. General Additive Rule Variants:** a) given $P(E), P(F)$ and that E and F are **independent**, then $P(E \cap F) = P(E)P(F)$ and so $P(E \cup F) = P(E) + P(F) - P(E)P(F)$.
b) Given $P(E), P(F)$ and that E and F are **mutually exclusive**, then $E \cap F = \emptyset$, so $P(E \cap F) = 0$ and so $P(E \cup F) = P(E) + P(F)$.

17) **Common Final Problem.** If a *fair coin* is tossed n times, then the probability of (exactly) r heads (or of r tails) is $\frac{C(n,r)}{2^n}$. S01 3b, S00 8, Q5 3, HW6 5, 6a, E2 9.

18) **Common Final Problem.** If a *fair coin* is tossed n times, find the *probability of at least j heads* or the *probability of at most j tails.* Let P(k) be the probability of k heads (or tails) in n tosses. Then P(**at least j** H's) = P(between j and n heads) $= P(j) + P(j+1) + \cdots + P(n) = 1 - P(0) - P(1) - \cdots - P(j-1)$. In particular,
P(**at least one** H) $= 1 - P(0) = 1 - 1/2^n$.
Similarly, P(**at most j heads**) = P(between 0 and j heads) $= P(0) + P(1) + \cdots + P(j) = 1 - P(j+1) - P(j+2) - \cdots - P(n)$. Notice that $P(j) = \frac{C(n,j)}{2^n}$. That is, use common final problem (11) repeatedly.

19) **Common Final problem:** Draw a tree diagram and use the tree to find the unconditional probability of a right branch event and use the tree and Bayes rule to find

a conditional probability of a left branch event given a right branch event. Probably $n = 3$ and it is not a jar problem. You will not be told to draw a tree diagram.

21) **Common Final problem:** Find the line $y = mx+b$ **perpendicular to the line** $Ax + By = C$ that contains the point (x_1, y_1). **Solution:** Two lines are perpendicular if the product of their slopes $= -1$. The given line has slope $-A/B$, hence $m(-A/B) = -1$ or $m = B/A$. Find b using the equation $y_1 = mx_1 + b$. That is, $b = y_1 - mx_1$. If the given line is already is slope intercept form, eg $y = m_1x + b_1$, then $m(m_1) = -1$, so $m = -1/m_1$.

22) **Common Final problem:** Find the horizontal and vertical lines through a point. **Solution:** The horizontal line through a point (x_1, y_1) is $y = y_1$ and the vertical line is $x = x_1$.

See S01 11e, Q8 1ab, Q9 1, HW11 71, 72 23) **Common Final problem:** Find the slope m of the line **parallel to the line** $Ax + By = C$. **Solution:** Parallel lines have the same slope, so $m = -A/B$.