Math 250 Exam 1 review. Thursday Feb. 5. Bring a TI–30 calculator but NO NOTES. Emphasis on sections 5.3, 5.5, 6.1, 6.2, 6.3, 3.7; HW1-5; Q1-4. Know for trig functions that $0.707 \approx \sqrt{2}/2$ and $0.866 \approx \sqrt{3}/2$.

From Math 150, for derivatives know power rule, product rule, quotient rule, chain rule and rules from reference p. 5. For integration know power rule, u-substitution and rules 1-20 from reference p. 6.

Types of problems from Math 150:

1) Be able to do basic integrals.

2) Be able to do integrals using u-substitution.

The following problems are **very important for exam 1 and the final**. The notation F*** means it was on 3 out of 3 of the last 3 finals.

For **integration by parts**, $\int udv = uv - \int vdu$ or
$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$ where
$u = f(x)$, $dv = g'(x)dx$,
$du = f'(x)dx$, $v = g(x) = \int g'(x)dx = \int dv$.

Notice that $v = g(x)$ is an antiderivative of $g'(x)$ with C = 0. Sometimes u-substitution is needed to find $v = g(x)$. Often you need to use integration by parts again (or u-substitution) to find $\int vdu = \int g(x)f'(x)dx$.

Consider the following list of functions:
1) logarithmic, 2) inverse trigonometric, 3) algebraic, 4) trigonometric, 5) exponential. The **LIATE principle for integration by parts** says try to choose a u as close to the beginning of the list as possible. Thus a log term will almost always be u while an exponential term will almost always be part of dv.

F1***) Indefinite integral using **integration by parts**: $\int udv = uv - \int vdu$ or $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$. F07–2b, S08–1a, F08–3c.

F2*) Definite integral using **integration by parts**: $\int_a^b udv = uv|_a^b - \int_a^b vdu$ or $\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx$. F07–1a.

F3**) Find an indefinite integral $\int \sin^m x \cos^n x\, dx$ or definite integral $\int_a^b \sin^m x \cos^n x\, dx$ where m or n is odd. If both terms are odd, use the method with $\min(m, n)$.

Know the technique for getting the final formula, not just the final formula.

a) $\int \sin^m x \cos^n x\, dx$ where the cos power $n = 2k + 1$ is odd.
Then pull out a $\cos x$ and use $\cos^2 x = 1 - \sin^2 x$ and u substitution with $u = \sin x$.
Then $\int \sin^m x \cos^n x\, dx = \int \sin^m x\ (1 - \sin^2 x)^k\ \cos x\, dx = \int u^m(1 - u^2)^k du$.

Special case: $m = 0$ where n is odd gives $\int \cos^n x\, dx = \int (1 - u^2)^k du$. F08 3a

b) $\int \sin^m x \cos^n x\, dx$ where the sin power $m = 2k + 1$ is odd.
Then pull out a $\sin x$ and use $\sin^2 x = 1 - \cos^2 x$ and u substitution with $u = \cos x$.
Then $\int \sin^m x\ \cos^n x\, dx = \int (1 - \cos^2 x)^k\ \cos^n x\ \sin x\, dx = -\int (1 - u^2)^k\ u^n\, du$.

Special case: $n = 0$ where m is odd gives $\int \sin^m x\, dx = -\int (1 - u^2)^k\, du$. F07 1b

F4*) Find an indefinite integral $\int \tan^m x \sec^n x \, dx$ or definite integral $\int_a^b \tan^m x \sec^n x \, dx$ where m is odd or n is even. S08–1b

Know the technique for getting the final formula, not just the final formula.

a) $\int \tan^m x \sec^n x \, dx$ where the sec power $n = 2k$ is even.
Then pull out a $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ and u substitution with $u = \tan x$.
Then $\int \tan^m x \sec^n x \, dx = \int \tan^m x \; (1 + \tan^2 x)^{k-1} \; \sec^2 x \, dx = \int u^m (1 + u^2)^{k-1} du$.

b) $\int \tan^m x \sec^n x dx$ where the tan power $m = 2k + 1$ is odd.
Pull out $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ and u substitution with $u = \sec x$.
Then $\int \tan^m x \sec^n x \, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx = \int (u^2 - 1)^k u^{n-1} du$.

If m is odd and n is even, use the method with $\min(m, n)$.

F5) a) $\int \cos^2 x \, dx = \int \frac{1}{2} + \frac{1}{2} \cos 2x \, dx$.

b) $\int \sin^2 x \, dx = \int \frac{1}{2} - \frac{1}{2} \cos 2x \, dx$.

F6) $\int \tan^m x \, dx$
Pull out $\tan^2 x = \sec^2 x - 1$ so $\tan^m x = \tan^{m-2} x \tan^2 x = [\tan^{m-2} x](\sec^2 x - 1) = [\tan^{m-2} x][\sec^2 x] - \tan^{m-2} x$.
Then repeat the process with $\tan^{m-2} x$. Eventually use u-substitution with $u = \tan x$.

F7) $\int \sec^n x \, dx$ for n even, especially $n = 4$.
Pull out $\sec^2 x = \tan^2 x + 1$ and use u-substitution with $u = \tan x$.

Know the technique for getting the final formula, not just the final formula.

So $\int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx = \displaystyle\int (\tan^2 x + 1)^{\frac{n-2}{2}} \sec^2 x \, dx = \int (u^2 + 1)^{\frac{n-2}{2}} \, du$.

F8) For $\int \sin^2 ax \, dx$, $\int \sin^3 ax \, dx$, $\int \cos^2 ax \, dx$, or $\int \cos^3 ax \, dx$, could use u-substitution with $u = ax$, then find $\frac{1}{a} \int \sin^m u \, du$ or $\frac{1}{a} \int \cos^n u \, du$.

Also $\sin^2 ax = \frac{1}{2} - \frac{1}{2} \cos 2ax$ and $\cos^2 ax = \frac{1}{2} + \frac{1}{2} \cos 2ax$.

For trigonometric substitution, the following triangle is useful.

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                        .
                   .    |
         hyp  .         |
             .          |   opp
          .             |
      .  theta          |
     __________________|   Recall the following identities.
              adj
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$\sin\theta = \frac{opp}{hyp}$ $\quad \csc\theta = \frac{hyp}{opp}$
$\cos\theta = \frac{adj}{hyp}$ $\quad \sec\theta = \frac{hyp}{adj}$
$\tan\theta = \frac{opp}{adj}$ $\quad \cot\theta = \frac{adj}{opp}$

If $x = a \sin\theta$, then $\sin\theta = x/a$. So $hyp = a$, $opp = x$ and $adj = \sqrt{a^2 - x^2}$.
If $x = a \tan\theta$, then $\tan\theta = x/a$. So $hyp = \sqrt{a^2 + x^2}$, $opp = x$ and $adj = a$.
If $x = a \sec\theta$, then $\sec\theta = x/a$ and $\cos\theta = a/x$.
So $hyp = x$, $opp = \sqrt{x^2 - a^2}$ and $adj = a$.

F9***) Trigonometric substitution is used to find integrals where the integrand contains $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$. The triangle below F8) will be labelled with a, x, and the $\sqrt{\ }$ from the integrand. Take $a > 0$. Get the indefinite integral in terms of θ, then use the triangle to convert back to x. F07 2c, S08 3, F08 2b.

a) If the integrand contains $\sqrt{a^2 - x^2}$,

then use $x = a \sin\theta$, $dx = a \cos\theta \; d\theta$ and $\sqrt{a^2 - x^2} = a\cos\theta$. To label the triangle with x, a and $\sqrt{a^2 - x^2}$, note that $\sin\theta = x/a$. Try to avoid using $\theta = \sin^{-1}(x/a)$.

b) If the integrand contains $\sqrt{a^2 + x^2}$,

then use $x = a\tan\theta$, $dx = a\sec^2\theta \; d\theta$ and $\sqrt{a^2 + x^2} = a\sec\theta$. To label the triangle with x, a and $\sqrt{a^2 + x^2}$, note that $\tan\theta = x/a$. Try to avoid using $\theta = \tan^{-1}(x/a)$.

c) If the integrand contains $\sqrt{x^2 - a^2}$,

then use $x = a\sec\theta$, $dx = a\sec\theta\tan\theta \; d\theta$ and $\sqrt{x^2 - a^2} = a\tan\theta$. To label the triangle with x, a and $\sqrt{x^2 - a^2}$, note that $\sec\theta = x/a$ so $\cos\theta = a/x$. Try to avoid using $\theta = \sec^{-1}(x/a)$.

Tips: i) the triangle can be used to convert $\sin\theta, \cos\theta, \tan\theta, \sec\theta, \csc\theta,$ and $\cot\theta$ terms to terms containing x.

ii) Know that $\cos^2\theta + \sin^2\theta = 1$, $\sec^2\theta = 1 + \tan^2\theta$,
$\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 1 - 2\sin^2\theta$,
$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$, $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$.

iii) Sometimes using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ or $\sec\theta = \dfrac{1}{\cos\theta}$ helps.

iv) $\int \tan\theta d\theta = \ln|\sec\theta| + C$ and $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$

v) $\int \sin 2\theta d\theta = \frac{-1}{2}\cos 2\theta + C$ and $\int \cos 2\theta d\theta = \frac{1}{2}\sin 2\theta + C$

vi) Note that $(a^2 + x^2)^{3/2} = [\sqrt{a^2 + x^2}\,]^3$ and $(a^2 + x^2) = [\sqrt{a^2 + x^2}\,]^2$

F10***) Partial fraction expansions are used to integrate a rational function $\int \frac{P(x)}{Q(x)} dx$ where $P(x)$ and $Q(x)$ are polynomials. Let the polynomial $R(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ where $a_n \neq 0$ and $n \geq 0$. Then the "degree of $R(x)$" = $\deg(R(x)) = n$. Note that $x = x^1$ and $1 = x^0$.

For the following 4 cases, terms of the form $a_i x + b_i$, where a_i and b_i are constants, could equivalently be written as $a_i x - b_i$ (or even $a_i x \pm b_i$).

case i) Each distinct linear factor of the form $a_i x + b_i$ in the denominator $Q(x)$ gives rise to a term $\dfrac{A_i}{a_i x + b_i}$ where the A_i are constants to be determined.

case ii) Each distinct factor of the form $(a_1 x + b_1)^r$ in the denominator $Q(x)$ gives rise to a term $\dfrac{A_1}{a_1 x + b_1} + \dfrac{A_2}{(a_1 x + b_1)^2} + \cdots + \dfrac{A_r}{(a_1 x + b_1)^r}$ where the $A_1, ..., A_r$ are constants to be determined. (Usually r = 2.)

case iii) Each distinct irreducible quadratic factor of the form $ax^2 + bx + c$ (with $b^2 - 4ac < 0$) in the denominator $Q(x)$ gives rise to a term $\dfrac{Ax + B}{ax^2 + bx + c}$ where A and B are constants to be determined.

case iv) Each distinct repeated irreducible quadratic factor of the form

$(ax^2+bx+c)^r$ (with $b^2-4ac<0$) in the denominator $Q(x)$ gives rise to a term $\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_rx+B_r}{(ax^2+bx+c)^r}$ where the A_i and B_i are constants to be determined for $i=1,...,r$. (This case rarely occurs on exams.)

Step 0) Check that $\deg(P(X))<\deg(Q(x))$. If the $\deg(P(X))\geq\deg(Q(x))$, use **long division** to write $\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$ where $S(x)$ and $R(x)$ are polynomials and $\deg(R(X))<\deg(Q(x))$.

If $\deg(P(X))<\deg(Q(x))$, take $S(x)\equiv 0$, and $R(x)=P(x)$.

Step 1) Factor the denominator $Q(x)$ as far as possible.

Step 2) Write $\frac{R(x)}{Q(x)}$ as a sum of partial fractions of the form $\frac{A}{(a_1x+b_1)^i}$ or $\frac{Ax+B}{(ax^2+bx+c)^j}$ using cases i)-iv) where typically the exponents $i=1$ and $j=1$.

Step 3) Once $\frac{R(x)}{Q(x)}=$ RHS, multiply both sides by $Q(x)$ so $R(x)=RHS\ Q(x)$. Then both sides are polynomials. Find the unknown coefficients in $RHS\ Q(x)$.

Step 4) Integrate $S(x)+RHS$.

Tips: i) Typically there will be two or three unknowns A,B and C.

ii) Try something similar to the following when possible. Suppose $\frac{R(x)}{Q(x)}=\frac{R(x)}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$.

Then $R(x)=A(x-b)(x-c)+B(x-a)(x-c)+C(x-a)(x-b)$.

Set $x=a$ to get $R(a)=A(a-b)(a-c)$ so $A=\frac{R(a)}{(a-b)(a-c)}$.

Set $x=b$ to get $R(b)=B(b-a)(b-c)$ so $B=\frac{R(b)}{(b-a)(b-c)}$.

Set $x=c$ to get $R(c)=C(c-a)(c-b)$ so $C=\frac{R(c)}{(c-a)(c-b)}$.

iii) Suppose $R(x)=ax+b$. Then solve $R(x)=RHS\ Q(x)$ for unknowns A and B by equating coefficients. So if $R(x)=ax+b=RHS\ Q(x)=(A+B)x+(A-B)$, then
$A+B=a$
$A-B=b$. Plug in $A=a-B$ into the second equation to get $a-B-B=b$ or $-2B=b-a$ or $B=(a-b)/2$. So $A=a-B=a-(a-b)/2$.

iv) Suppose $R(x)=a^2x^2+bx+c$. Then solve $R(x)=RHS\ Q(x)$ for unknowns A, B and C by equating coefficients. Typically one or two of the resulting 3 equations needs to be "easy."

If $R(x)=1+x=0x^2+x+1=(A+B)x^2+Cx+A$, then
$A+B=0$
$C=1$
$A=1$ so $B=-A=-1$.

If $R(x)=-2x=0x^2-2x+0=(A+B)x^2+(B+C)x+(A+C)$, then
1) $A+B=0$
2) $B+C=-2$
3) $A+C=0$

So by 3) $A=-C$ and by 1) $A=-B=-C$ and $B=C$. So by 2), $2C=-2$ or $C=B=-1$. Thus $A=-B=1$.

v) $\int\frac{b}{x-a}dx=b\ln|x-a|+C$.

vi) $\int\frac{b}{x^2+a^2}dx=\frac{b}{a}\tan^{-1}(\frac{x}{a})+C$.

vii) $\int\frac{bx}{x^2+a^2}dx=\frac{b}{2}\ln|x^2+a^2|+C$.

Let $\lim_{x\to a^*} f(x)$ denote $\lim_{x\to a} f(x)$, $\lim_{x\to a^-} f(x)$, or $\lim_{x\to a^+} f(x)$. Here $a = \pm\infty$ is allowed.

As shorthand notation, the indeterminant forms are $\frac{0}{0}, \frac{\infty}{\infty}, 0(\infty), \infty - \infty, 0^0, \infty^0$ and 1^∞. For $\frac{\infty}{\infty}$, the sign could be $\pm\infty$.

F11***) Limits using L'Hospital's rule: If $\lim_{x\to a^*} \frac{f(x)}{g(x)}$ has indefinite form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a^*} \frac{f(x)}{g(x)} = \lim_{x\to a^*} \frac{f'(x)}{g'(x)}$ if the limit of the RHS exists or is ∞ or is $-\infty$.
F07 3a, S08 6a, F08 1ab.

Tips: i) Sometimes L'Hospital's rule needs to be applied several times before the limit can be found (eg if $\lim_{x\to a^*} \frac{f'(x)}{g'(x)}$ is of form $\frac{0}{0}$, apply L'Hospital's rule again).
ii) $\lim_{x\to a^*} f(x) = f(a)$ for continuous functions.
iii) Know the limits of the trig functions for $a = 0, \pi/2$, and π.
iv) Make sure the limit is of indeterminant form.
v) Keep the symbol "$\lim_{x\to a^*}$" until you evaluate the limit (often using tip ii)).

F12***) Applying L'Hospital's rule for indeterminant forms $0(\infty), \infty - \infty, 0^0, \infty^0$ or 1^∞. Transform to a limit with indeterminant form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then use F11).
F07 3b, S08 6b, F08 1cd.

a) If $\lim_{x\to a^*} f(x)g(x)$ has indeterminant form $0(\infty)$, write $f(x)g(x) = \frac{f(x)}{1/g(x)}$ or $f(x)g(x) = \frac{g(x)}{1/f(x)}$ using the option that gives the simplest limit.

b) If $\lim_{x\to a^*} \frac{h(x)}{f(x)} - \frac{k(x)}{g(x)}$ has indeterminant form $\infty - \infty$, then write $\frac{h(x)}{f(x)} - \frac{k(x)}{g(x)} = \frac{h(x)g(x) - k(x)f(x)}{f(x)g(x)}$.

c) If $\lim_{x\to a^*} f(x)^{g(x)}$ has indeterminant form $0^0, \infty^0$ or 1^∞, evaluate $\lim_{x\to a^*} \ln f(x)^{g(x)} = \lim_{x\to a^*} g(x) \ln f(x) = \lim_{x\to a^*} \frac{\ln f(x)}{\frac{1}{g(x)}} = L$ and use $\lim_{x\to a^*} f(x) = e^L$.
Note that $f(x)^{g(x)} > 0$ is needed since $\ln(x)$ is defined for $x > 0$. Note that if $g(x) = 1/k(x)$, then $k(x) = 1/g(x)$.

Tips: i) Use the following trick: $\lim_{x\to 0+} \sqrt{x}\ln x = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{\sqrt{x}}}$.

ii) Try to simplify the power on x. So $\lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} -x^2x^{-1} = \lim_{x\to 0+} -x = 0$.

Similarly, $\lim_{x\to 0+} \frac{1/x}{(\frac{-1}{2})(x^{-3/2})} = \lim_{x\to 0+} -2x^{3/2}x^{-1} = \lim_{x\to 0+} -2\sqrt{x} = 0\ (= L)$.