Math 250 Exam 2 review. Thursday March 5. Bring a TI–30 calculator but NO NOTES. Emphasis on sections 5.5, 6.1, 6.2, 6.3, 3.7, 6.6, 8.1, 8.2, 8.3, part of 8.4; HW1-12; Q1-11. Know for trig functions that $0.707 \approx \sqrt{2}/2$ and $0.866 \approx \sqrt{3}/2$.

From Math 150, for derivatives know power rule, product rule, quotient rule, chain rule and rules from reference p. 5. For integration know power rule, u-substitution and rules 1-20 from reference p. 6.

Know everything from Exam 1 review, including F1)-F12).

The following problems are **very important for exam 2 and the final**. The notation F*** means it was on 3 out of 3 of the last 3 finals.

F13*) Occasionally you need to find a limit using Math 150 techniques instead of L'Hospital's rule. F08 1b

An **improper integral** is $\int_a^b f(x)dx$ where $a = -\infty$, $b = \infty$ or where $f(x)$ has a vertical asymptote for some $c \in [a, b]$.

Improper integrals are defined as limits. If the limit exists as a real number, then $\int_a^b f(x)dx$ is **convergent**. If the limit does not exist, is ∞ or is $-\infty$, then $\int_a^b f(x)dx$ is **divergent**.

i) $\displaystyle\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx,$ ii) $\displaystyle\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx.$

Assume the two integrals on the RHS below are convergent. Let a be any constant (chosen so that you can evaluate the two RHS integrals), then

iii) $\displaystyle\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx = I_1 + I_2.$

For iii), if either I_1 or I_2 is divergent, then $\int_{-\infty}^\infty f(x)dx$ is divergent. **So stop as soon as one of the two integrals is shown to be divergent.**

Now let $a < b$ be real and suppose $f(x)$ is continuous on $[a, b]$ except for a vertical asymptote at a, b or $c \in (a, b)$.
(Several vertical asymptotes can be handled in a similar manner.)

iv) If the vertical asymptote is at b, then $\displaystyle\int_a^b f(x)dx = \lim_{t\to b-}\int_a^t f(x)dx.$

v) If the vertical asymptote is at a, then $\displaystyle\int_a^b f(x)dx = \lim_{t\to a+}\int_t^b f(x)dx.$

vi) If the vertical asymptote is at c, then $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx =$ $\displaystyle\lim_{t\to c-}\int_a^t f(x)dx + \lim_{t\to c+}\int_t^b f(x)dx = I_1 + I_2.$ Again, if either I_1 or I_2 is divergent, then $\int_{-\infty}^\infty f(x)dx$ is divergent. **So stop as soon as one of the two integrals is shown to be divergent.**

Be able to show that $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \text{is divergent}, & p \le 1. \end{cases}$

Know that $\int_1^\infty \frac{1}{x^p}\,dx$ converges iff $p > 1$, that $\int_0^1 \frac{1}{x^p}\,dx$ converges iff $p < 1$, and $\int_0^\infty \frac{1}{x^p}\,dx$ diverges for all real p. Hence $\int_{-\infty}^\infty \frac{1}{x^p}\,dx$ diverges for all real p. **But you need to show convergence or divergence by taking limits for exam, quiz and homework problems.**

Comparison Theorem: Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

i) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

ii) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Know what the graphs of $\tan t$, $\sec t$ (and other trig functions) look like on ref. p. 2.
Know what the graph of $\ln t$, e^t and e^{-t} look like (see p. 144, 146, 154 and ref. p. 4).
Know what the graph of $\tan^{-1} t$ looks like on ref. p. 3.

Memorize the following limits.

$\lim_{t\to\infty} e^t = \infty$, $\lim_{t\to-\infty} e^t = 0$

$\lim_{t\to\infty} e^{-t} = 0$, $\lim_{t\to-\infty} e^{-t} = \infty$

$\lim_{t\to\infty} \ln t = \infty$, $\lim_{t\to 0+} \ln t = -\infty$

$\lim_{t\to\frac{\pi}{2}-} \tan t = \infty$, $\lim_{t\to\frac{\pi}{2}+} \tan t = -\infty$

$\lim_{t\to\frac{\pi}{2}-} \sec t = \infty$, $\lim_{t\to\frac{\pi}{2}+} \sec t = -\infty$

$\lim_{t\to\infty} \tan^{-1} t = \pi/2$, $\lim_{t\to-\infty} \tan^{-1} t = -\pi/2$

$\lim_{t\to\infty} \sec^{-1} t = \pi/2$, $\lim_{t\to-\infty} \sec^{-1} t = \pi/2$

F14)** Show $\int_a^b f(x)dx$ or $\int_a^\infty f(x)dx$ or $\int_{-\infty}^b f(x)dx$ is divergent because i) the limit is $\pm\infty$ or does not exist, or ii) by using the Comparison Theorem. F07 5, S08 5,

Integrals $\displaystyle\int_{-a}^a \frac{dx}{x^p}$ and $\displaystyle\int_{-a}^a \frac{(p-1)dx}{x^p}$ are especially common where $p > 1$ and often $a = 1$.

F15*)** Find $\int_a^b f(x)dx$ for an improper integral where $a = -\infty$ and $b = \infty$ are allowed. F07 4, S08 4, F08 3c.

Often the integral needs to be found using trig substitution or

i) $\displaystyle\lim_{t\to\infty}\int_c^t \frac{dx}{a^2+x^2} = \lim_{t\to\infty}\frac{1}{a}[\tan^{-1}(\frac{t}{a}) - \tan^{-1}(\frac{c}{a})] = \frac{1}{a}[\frac{\pi}{2} - \tan^{-1}(\frac{c}{a})]$ or

ii) $\displaystyle\lim_{t\to\infty}\int_c^t \frac{dx}{x\sqrt{x^2-a^2}} = \lim_{t\to\infty}\frac{1}{a}[\sec^{-1}(\frac{t}{a}) - \sec^{-1}(\frac{c}{a})] = \frac{1}{a}[\frac{\pi}{2} - \sec^{-1}(\frac{c}{a})]$.

(Often $a = 1$. For i), often c = 0. For ii) need $c > 1$ and often $c = 2a$.)

A sequence $\{a_n\} = \{a_n\}_1^\infty = \{a_1, a_2, a_3 ...\}$ where there is a function $a(n) = a_n$ with domain the positive integers $\{1, 2, 3, ...\}$.

If $\lim_{n\to\infty} a_n = L$, a real number, then the sequence $\{a_n\}$ converges or is convergent. If the limit does not exist or is $\pm\infty$, then the sequence $\{a_n\}$ diverges or is divergent.

Notation for sums is useful. $\displaystyle\sum_{n=1}^\infty a_n = a_1 + a_2 + a_3 + \cdots$.

$\displaystyle\sum_{n=1}^3 a_n = a_1 + a_2 + a_3$.

Integrals are limits of sums: $\displaystyle\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\frac{b-a}{n}$ where $x_i = a + i(b-a)/n$.

Algebra of sums: $\displaystyle\sum_{i=1}^n 1 = n$, $\displaystyle\sum_{i=1}^n c = cn$, $\displaystyle\sum_{i=1}^n ca_i = c\sum_{i=1}^n a_i$, $\displaystyle\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$.

An **infinite series** is a sum $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$.

The nth partial sum $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$.

If the sequence $\{s_n\}$ is convergent with $\lim_{n\to\infty} s_n = s$, a real number, then $\sum_{n=1}^{\infty} a_n$ is convergent or converges with $\sum_{n=1}^{\infty} a_n = s$. If $\{s_n\}$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent or diverges.

Suppose $a \neq 0$. The **geometric series** $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $|r| < 1$. If $|r| \geq 1$, then the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

A convergent infinite series remains convergent if the 1st k terms are discarded or if $k+1$ additional terms $a_0, a_{-1}, ..., a_{-k}$ are added. A divergent infinite series remains divergent if the 1st k terms are discarded or if $k+1$ additional terms are added.

Note that the geometric series $\sum_{n=0}^{\infty} ar^{n-1} = a_0 + \sum_{n=1}^{\infty} ar^{n-1}$.

Note that $\sum_{n=1}^{\infty} d\ r^n = \sum_{n=1}^{\infty} dr\ r^{n-1}$ is a geometric series with $a = dr$.

Suppose $\sum_{n=1}^{\infty} a_n$ is convergent (divergent). Then $c \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ca_n$ remains convergent (divergent) for any $c \neq 0$..

The **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ converges for $p > 1$ and diverges for $p \leq 1$. The divergent 1 series is also called the harmonic series.

Let n and d be integers with $d \geq 1$. The **telescoping series**
$\sum_{n=1}^{\infty} \frac{kd}{(n+c)(n+d+c)} = k \sum_{n=1}^{\infty} \frac{d}{(n+c)(n+d+c)} = k \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{1}{i+c} - \frac{1}{i+d+c}) =$
$k \lim_{n\to\infty} [\frac{1}{1+c} + \frac{1}{2+c} + \cdots + \frac{1}{d+c} - \frac{1}{n+1+c} - \cdots - \frac{1}{n+d+c}] =$
$k[\frac{1}{1+c} + \cdots + \frac{1}{d+c}]$.

Similarly, the telescoping series
$\sum_{n=j+1}^{\infty} \frac{kd}{(n+c)(n+d+c)} = k \sum_{n=j+1}^{\infty} \frac{d}{(n+c)(n+d+c)} = k \lim_{n\to\infty} \sum_{i=j+1}^{n} (\frac{1}{i+c} - \frac{1}{i+d+c}) =$
$k \lim_{n\to\infty} [\frac{1}{j+1+c} + \frac{1}{j+2+c} + \cdots + \frac{1}{j+d+c} - \frac{1}{n+1+c} - \cdots - \frac{1}{n+d+c}]$
$= k[\frac{1}{j+1+c} + \cdots + \frac{1}{j+d+c}]$.

Typically $k = 1$ or $k = 1/d$, $j = 1$ and $1 \leq d \leq 3$. To see this claim, note that

the partial sum $s_n = \sum_{i=1}^{n} \frac{kd}{(i+c)(i+d+c)}$. The term a_i has partial fraction expansion $\frac{A}{i+c} + \frac{B}{i+d+c}$ or $kd = A(i+c+d) + B(i+c)$. If $i = -c$, than $kd = Ad$ or $A = k$. If $i = -(c+d)$, then $kd = B(-d)$ or $B = -k$. Thus $s_n = \sum_{i=1}^{n}(\frac{k}{i+c} - \frac{k}{i+d+c})$. Thus $\sum_{n=1}^{\infty} \frac{kd}{(n+c)(n+d+c)} = \lim_{n\to\infty} s_n = k \lim_{n\to\infty} \sum_{i=1}^{n}(\frac{1}{i+c} - \frac{1}{i+d+c})$. All but the 1st d terms of $\frac{1}{i+c}$ and the last d terms of $\frac{-1}{i+d+c}$ cancel when s_n is written as a telescoping sum. Thus $\sum_{n=1}^{\infty} \frac{kd}{(n+c)(n+d+c)} =$
$k \lim_{n\to\infty}[\frac{1}{1+c} + \frac{1}{2+c} + \cdots + \frac{1}{d+c} - \frac{1}{n+1+c} - \cdots - \frac{1}{n+d+c}]$.

When finding $\sum_{n=1}^{\infty} \frac{kd}{(n+c)(n+d+c)}$, **write out the partial fraction expansion to find** A and B and find the limit of the partial sum $\lim_{n\to\infty} k \sum_{i=1}^{n}(\frac{1}{i+c} - \frac{1}{i+d+c}) =$
$k \lim_{n\to\infty}[\frac{1}{1+c} + \frac{1}{2+c} + \cdots + \frac{1}{d+c} - \frac{1}{n+1+c} - \cdots - \frac{1}{n+d+c}]$.

You should be able to find the $2d$ terms of $s_n = \sum_{i=j+1}^{n}(\frac{1}{i+c} - \frac{1}{i+d+c})$ quickly since they are the 1st d terms of $\frac{1}{i+c}$ and the last d terms of $\frac{-1}{i+d+c}$.

Ex. $\sum_{i=2}^{n}(\frac{1}{i-1} - \frac{1}{i+1}) = \frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$ since $d = 2$ with $i+1 = i-1+d = i-1+2$.

Ex. $\sum_{i=1}^{n}(\frac{1}{i} - \frac{1}{i+3}) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$ since $d = 3$.

The nth term test for divergence: If it is not true than $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

So $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n = L \neq 0$ or if $\lim_{n\to\infty} a_n$ does not exist. Note that $\lim_{n\to\infty} a_n = 0$ is inconclusive: the series $\sum_{n=1}^{\infty} a_n$ could converge or diverge.

The integral test: Suppose f is a continuous, positive, decreasing function on $[c, \infty)$. Let $a_n = f(n)$.

a) If $\int_c^{\infty} f(x)dx$ is convergent, then $\sum_{n=c}^{\infty} a_n$ is convergent.

b) If $\int_c^{\infty} f(x)dx$ is divergent, then $\sum_{n=c}^{\infty} a_n$ is divergent.

Tips: i) Usually $c = 1$ and the sum need not start at c since convergence or divergence of an infinite series $\sum_{i=d}^{\infty} a_n$ does not depend on the 1st k terms.

ii) Be able to show that $\int_1^\infty \frac{1}{x^p}\ dx = \frac{1}{p-1}$ if $p > 1$ to show that the p series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges for $p > 1$.

iii) Be able to show that $\int_1^\infty \frac{1}{x^p}\ dx = \infty$ if $p \le 1$ to show that the p series $\sum_{n=1}^\infty \frac{1}{n^p}$ diverges for $p \le 1$.

iv) If $\int_1^\infty f(x)\ dx = L$, the series $\sum_{n=1}^\infty a_n$ converges, but $\sum_{n=1}^\infty a_n \ne L$ in general.

v) The integral test is often used where $f(x) = 1/g(x)$ or $f(x) = \frac{1}{g(x)h(x)}$ where $g(x)$ and $h(x)$ are increasing.

vi) Recall that continuous $f(x)$ is decreasing on $[c,\infty)$ if $f'(x) < 0$ on (c,∞).

vii) Function $g(x) = x^p$ is positive and increasing for $x > 0$ and $p > 0$, $h(x) = [\ln(x)]^p$ is positive and increasing increasing for $x > 1$ and $p > 0$, $g(x) = e^x$ is positive and increasing on $(-\infty,\infty)$.

viii) The function $g(x) = \ln(x)/x$ is positive and decreasing for $x > e$ so for $x > 3$.

Know that i) $\frac{x^p}{e^x} \to 0$ as $x \to \infty$ for all p.

So ii) $\frac{e^x}{x^p} \to \infty$ as $x \to \infty$ for all p.

iii) $\frac{\ln x}{x} \to 0$ as $x \to \infty$.

So iv) $\frac{x}{\ln x} \to \infty$ as $x \to \infty$.

Let $\tan^{-1}(\theta) = w$. Then $\tan[\tan^{-1}(\theta)] = \theta = \tan w$ and w is the angle (in radians) whose tan is θ. So find $w \in (-\pi/2, \pi/2)$ such that $\tan(w) = \theta$. Then $\tan^{-1}(\theta) = w$. Know that $\tan(0) = 0$ and $\tan(\pi/4) = 1$. Thus $\tan^{-1}(0) = 0$ and $\tan^{-1}(1) = \pi/4$.

Let $\sin^{-1}(\theta) = w$. Then $\sin[\sin^{-1}(\theta)] = \theta = \sin w$ and w is the angle (in radians) whose sine is θ. So find $w \in [-\pi/2, \pi/2]$ such that $\sin(w) = \theta$. Then $\sin^{-1}(\theta) = w$. Know that $\sin(0) = 0$, $\sin(\pi/2) = 1$ and $\sin(-\pi/2) = -1$. Thus $\sin^{-1}(0) = 0$, $\sin^{-1}(1) = \pi/2$, and $\sin^{-1}(-1) = -\pi/2$.

Let $\sec^{-1}(\theta) = w$. Then $\sec[\sec^{-1}(\theta)] = \theta = \sec w = 1/\cos(w)$. So find $w \in (0, \pi/2) \cup (\pi/2, \pi)$ such that $\sec(w) = \theta$ or $\cos(w) = 1/\theta$. Then $\sec^{-1}(\theta) = w$. Know that $\cos(\frac{\pi}{3}) = 1/2$ so $2 = \sec(\pi/3)$. Thus $\sec^{-1}(2) = \pi/3$.

The Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

a) If $\sum b_n$ is convergent and $0 < a_n \le b_n$ for all n, then $\sum a_n$ is convergent.

b) If $\sum b_n$ is divergent and $0 < b_n \le a_n$ for all n, then $\sum a_n$ is divergent.

Tips: For the comparison test, usually $\sum b_n = \sum \frac{1}{n^p}$ is a p series which converges for $p > 1$ and diverges for $p \le 1$, or $\sum b_n = \sum a\ r^{n-1}$ is a geometric series which converges for $|r| < 1$ and diverges for $|r| \ge 1$.

The Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$ where c is real, then either both series diverge or both series converge.

Note: "n!" is read "n factorial" and $n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$. So $n! = n[(n-1)!] = n(n-1)[(n-2)!] = n(n-1)(n-2)[(n-3)!]$ et cetera. Here $n \geq 1$ is an integer and $0! = 1$.

Know that n^n dominates $n!$ which dominates the exponential c^n for any constant c. Also the exponential c^n dominates n^p for any $c > 1$, $p > 0$. Thus $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0 = \lim\limits_{n\to\infty} \dfrac{n}{n}\,\dfrac{n-1}{n}\,\dfrac{n-2}{n}\cdots\dfrac{3}{n}\,\dfrac{2}{n}\,\dfrac{1}{n}$. Similarly, $\lim\limits_{n\to\infty} \dfrac{c^n}{n^n} = 0 = \lim\limits_{n\to\infty} \dfrac{c}{n}\,\dfrac{c}{n}\,\dfrac{c}{n}\cdots\dfrac{c}{n}\,\dfrac{c}{n}$ and $\lim\limits_{n\to\infty} \dfrac{n^p}{n^n} = 0 = \lim\limits_{n\to\infty} \dfrac{1}{n^{n-p}}$. Now $\lim\limits_{n\to\infty} \dfrac{c^n}{n!} = 0 = \lim\limits_{n\to\infty} \dfrac{c}{n}\,\dfrac{c}{n-1}\,\dfrac{c}{n-2}\cdots\dfrac{c}{2}\,\dfrac{c}{1}$. By L'Hospital's rule, $\lim\limits_{n\to\infty} \dfrac{n^p}{c^n} = \lim\limits_{n\to\infty} \dfrac{p\ n^{p-1}}{\ln(c)\ c^n} = \lim\limits_{n\to\infty} \dfrac{p(p-1)\ n^{p-2}}{[\ln(c)]^2\ c^n} = \lim\limits_{n\to\infty} \dfrac{p(p-1)(p-2)\ n^{p-3}}{[\ln(c)]^3\ c^n}$ $= \cdots = \lim\limits_{n\to\infty} \dfrac{p!}{[\ln(c)]^p\ c^n} = 0$ if $c > 1$ and $p \geq 1$ is an integer. Thus $\lim\limits_{n\to\infty} \dfrac{n^p}{c^n} = 0$ if $c > 1$ for any real p by the comparison theorem.

To summarize, $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$, $\lim\limits_{n\to\infty} \dfrac{c^n}{n^n} = 0$ for any constant c, $\lim\limits_{n\to\infty} \dfrac{n^p}{n^n} = 0$ and
$\lim\limits_{n\to\infty} \dfrac{n^p}{n!} = 0$ for any constant p, $\lim\limits_{n\to\infty} \dfrac{c^n}{n!} = 0$ for any constant c, and
$\lim\limits_{n\to\infty} \dfrac{n^p}{c^n} = 0$ for any $c > 1$ and any real p.

Also, know that $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$.

An **alternating series** has terms that are alternately positive and negative. If $b_n > 0$, then $\sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ is an alternating series where the first term $a_1 = b_1$ is positive, while $\sum\limits_{n=1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$ is an alternating series where the first term $a_1 = -b_1$ is negative.

Alternating Series Test: Suppose $b_n > 0$ and $\sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n$ (or $\sum\limits_{n=1}^{\infty} (-1)^n b_n$) is an alternating series.

i) If $b_{n+1} \leq b_n$ for all n (or if b_n is decreasing), and ii) $\lim\limits_{n\to\infty} b_n = 0$,
then the series is convergent.

Tip: For an alternating series, try the nth term test for divergence and the Alternating Series Test for convergence. Note that if $\lim\limits_{n\to\infty} b_n \neq 0$ then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (-1)^{n-1} b_n \neq 0$ (does not exist), and $\sum a_n$ diverges by the nth term test.

A series $\sum_n^\infty a_n$ is **absolutely convergent** if $\sum_n^\infty |a_n|$ is convergent.

A series $\sum_n^\infty a_n$ is **conditionally convergent** if $\sum_n^\infty |a_n| = \infty$ is divergent but $\sum_n^\infty a_n$ is convergent.

Absolute Convergence Theorem: If $\sum_n^\infty a_n$ is absolutely convergent, then $\sum_n^\infty a_n$ is convergent.

Thus if $\sum_n^\infty a_n$ is divergent then $\sum_n^\infty a_n$ is not absolutely convergent and $\sum_n^\infty |a_n| = \infty$ is divergent. If $\sum_n^\infty |a_n|$ is convergent, then $\sum_n^\infty a_n$ is convergent.

Comparison Test for Absolute Convergence: If $|a_n| \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ is absolutely convergent and so convergent.

Tip: This test is useful for showing absolute convergence, and is often useful if the numerator equals (or contains) $\sin(n)$, $\cos(n)$ or $(-1)^n$ since then the $|\sin(n)| \leq 1$, $|\cos(n)| \leq 1$ and $|(-1)^n| = 1 \leq 1$.

The Ratio Test: i) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then $\sum a_n$ is absolutely convergent and so convergent.

ii) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$ where $L = \infty$ is allowed, then $\sum a_n$ divergent and so not absolutely convergent.

iii) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, then the test is inconclusive (so it is possible that $\sum a_n$ is divergent, conditionally convergent or absolutely convergent).

Tips: i) If the index n of a_n is invoved as an exponential or factorial (c^n or $(n-c)!$ where c is real), then the ratio test is often useful for showing convergence or divergence of $\sum a_n$.

ii) If n is **only** involved algebraically or logarithmically (eg $\sum \frac{1}{n^p}$, $\sum \frac{n}{(4n-3)(4n-1)}$ or $\sum \frac{\sqrt{2n-1}\ln(4n+1)}{n(n+1)}$), then often the ratio test will fail (be inconclusive).

The Root Test: i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is absolutely convergent and so convergent.

ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ where $L = \infty$ is allowed, then $\sum a_n$ divergent and so not absolutely convergent.

iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the test is inconclusive (so it is possible that $\sum a_n$ is divergent, conditionally convergent or absolutely convergent).

Tips: i) The root test is useful when nth powers occur.

ii) $\lim_{n\to\infty} n^{1/n} = 1$ and $\lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$.

iii) $\sqrt[n]{|a_n|} = |a_n|^{1/n}$.

F16*)** Show that a series $\sum a_n$ converges or diverges using series tests. F08 4, S08 7, F07 6.

F17*)** Show that a series $\sum a_n$ is absolutely convergent, conditionally convergent or divergent. (Usually the series is a conditionally convergent alternating series, so you have to show that $\sum |a_n|$ is divergent, but $\sum a_n$ is convergent.) F08 5, S08 8, F07 7.

Tips: i) The **geometric series** $\sum_{n=j+1}^{\infty} ar^{n-1}$ and the **scaled geometric series** $\sum_{n=j+1}^{\infty} ar^{n+k}$ converge if $|r| < 1$ and diverge if $|r| \geq 1$.

ii) The **p-series** $\sum_{n=j+1}^{\infty} \frac{1}{n^p}$ and the **scaled p-series** $\sum_{n=j+1}^{\infty} \frac{c}{n^p}$ converge for $p > 1$ and diverge for $p \leq 1$.

iii) **The nth term (divergence) test:** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

This test is often useful for alternating series and for series where a_n is a ratio of polynomials and roots.

iv) **The integral test:** Suppose f is a continuous, positive, decreasing function on $[c, \infty)$. Let $a_n = f(n)$. Then $\sum_{n=c}^{\infty} a_n$ converges iff $\int_c^{\infty} f(x)dx$ is convergent.

The integral test is useful for p-series $\sum \frac{1}{n^p}$ and $\sum \frac{1}{n[\ln(n)]^p}$ and sometimes for functions $f(x) = 1/g(x)$ or $f(x) = 1/[g(x)h(x)]$ where g and h are increasing functions.

v) **Absolute Convergence Theorem:** If $\sum_{n=j+1}^{\infty} |a_n|$ converges, then $\sum_{n=j+1}^{\infty} a_n$ converges.

vi) **The Comparison Test**: a) If $\sum b_n$ converges and $0 \leq a_n \leq b_n$ for all n, then $\sum a_n$ converges. b) If $\sum b_n$ is diverges and $0 \leq b_n \leq a_n$ for all n, then $\sum a_n$ is diverges.

vii) **Comparison Test for Absolute Convergence:** If $|a_n| \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ is absolutely convergent and so convergent.

viii) **The Limit Comparison Test**: Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ where c is real, then either both series diverge or both series converge.

Tests vi)–viii) are useful if a_n is a ratio of polynomials and roots. For the limit comparison test, let $b_n = 1/n^p$ = largest power in numerator/largest power in denominator of a_n where $p > 0$ (so that the series converges).

ix) **Alternating Series Test:** Suppose $b_n > 0$ and $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ (or $\sum_{n=1}^{\infty}(-1)^n b_n$) is an alternating series. If a) $b_{n+1} \leq b_n$ for all n (or if b_n is decreasing), and b) $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

x) **The Ratio Test**: Let $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$. a) If $L < 1$, then $\sum a_n$ is absolutely convergent. b) If $L > 1$, then $\sum a_n$ diverges. c) If $L = 1$, then the test fails.

This test is useful if a_n contains terms like $(n + j)!$ and c^n.

xi) **The Root Test**: Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. a) If $L < 1$, then $\sum a_n$ is absolutely convergent. b) If $L > 1$, then $\sum a_n$ diverges. c) If $L = 1$, then the test fails.

This test is useful if $a_n = (c_n)^n$, $a_n = \frac{(c_n)^n}{n^p}$ or $a_n = \frac{n^p}{(c_n)^n}$.