Exam 1 is Th. Feb. 8. **You are allowed 4 sheets of notes and a calculator.**

The notation $a(t)$ is for the *accumulated value* (AV) at time t for an investment of 1 made at time 0.

$A(t)$ is the accumulated value at time t for an investment of $X = A(0)$ made at time 0. Then $A(t) = A(0)a(t) = Xa(t)$.

1) Given $A(t)$ be able to find $A(0)$, $a(0)$ and $a(t)$.

The *effective rate of interest* in year t is

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)} = \frac{A(t) - A(t-1)}{A(t-1)}.$$

2) Given $A(t)$ or $a(t)$, be able to find i_t.

A useful formula is $a(t) = \prod_{j=1}^{t}(1 + i_j) = (1 + i_1)(1 + i_2) \cdots (1 + i_t)$.

3) For **compound interest** $i_t \equiv i$ is a constant, and $a(t) = (1 + i)^t$ where i is the interest rate.

4) For *simple interest*, $a(t) = 1 + it$ where i is the simple interest rate, and $i_t = \dfrac{i}{1 + i(t-1)}$. Interest accumulated in a given year does not earn interest in future years.

The AV $A(t)$ of a fund is how much the fund is worth at time t if $A(0)$ is invested at time 0. The *present value* $PV = PV(t) = A_t(0) = A(0)$ is the amount invested in a fund at time 0 that will be worth $A(t)$ in t years = price of the investment made now.

The present value of 1 in t years is $PV = \dfrac{1}{a(t)}$.

5) For compound interest, $PV = \dfrac{1}{a(t)} = \dfrac{1}{(1+i)^t} = (1+i)^{-t} = v^t$ where $v = \dfrac{1}{1+i} = (1+i)^{-1}$.

Typically $i > 0$ and $0 < v < 1$.

6) **Know:** Let $PV = X = A(0)$ and $Y = A(t)$. For compound interest, let $A(0)(1+i)^t = X(1+i)^t = PV(1+i)^t = Y = A(t)$. Given 3 of $PV = X = A(0)$, $Y = A(t)$, i and t be able to find the 4th quantity.

i) $A(t) = A(0)(1 + i)^t = PV(1 + i)^t = PVv^{-t}$.

ii) $PV = A(0) = A(t)(1 + i)^{-t} = A(t)v^t$.

iii) $t = \dfrac{\ln[Y/X]}{\ln(1+i)} = \dfrac{\ln[A(t)/A(0)]}{\ln(v^{-1})}$.

iv) $i = \left(\dfrac{Y}{X}\right)^{1/t} - 1 = \left(\dfrac{A(t)}{A(0)}\right)^{1/t} - 1$.

On a BA II Plus, N = t, $I/Y = i$ (in percent), PV = PV and $FV = AV = A(t)$. For a lender (bank), *cash flow out* is a loan and *cash flow in* is a repayment of the loan. Cash flow out has a negate value. So the calculator expects one of PV and FV to be negative. Using PV negative seems to work.

7) A nominal rate of interest could be compounded semiannually (twice a year), quarterly (4 times a year), monthly (12 times a year) or every 2 years ("1/2 time" a year). Let $i^{(m)}$, read "i upper m", be the nominal annual rate of interest compounded

m times a year. Want to convert the nominal rate to an effective annual rate i. Then $\left(1 + \dfrac{i^{(m)}}{m}\right)^m = 1 + i$ and $i^{(m)} = m[(1+i)^{1/m} - 1]$. Note that $A(t) = A(0)\left(1 + \frac{i^{(m)}}{m}\right)^{mt}$.

The interpretation is that the effective interest rate for $\dfrac{1}{m}$th of a year is $i^{(m)}/m$.

Warning: HW and exam questions often use i for $i^{(m)}$.

The *effective rate of discount* in year t is

$$d_t = \frac{a(t) - a(t-1)}{a(t)} = \frac{A(t) - A(t-1)}{A(t)},$$

but usually find d_t as a function of i_t. Note that i_t has $a(t-1)$ in the denominator.

8) $v = \dfrac{1}{1+i} = 1 - d, \quad i = \dfrac{d}{1-d}, \quad d = \dfrac{i}{1+i} = iv, \quad i - d = id$, and $\dfrac{1}{d} - \dfrac{1}{i} = 1$.

9) Let $d^{(m)}$, read "d upper m", be the nominal annual rate of discount compounded m times a year. Want to convert the nominal rate to an effective annual rate d. Then $\left(1 - \dfrac{d^{(m)}}{m}\right)^m = 1 - d$ and $d^{(m)} = m[1 - (1-d)^{1/m}]$. Note that $A(t) = A(0)\left(1 - \frac{d^{(m)}}{m}\right)^{-mt}$.

10) **Memorize:** $A(t) = (PV)v^{-t}$, $PV = A(t)v^t$, and $v = 1 - d = (1+i)^{-1}$. Students often make a sign error in the exponent for points 11) and 12) below, and these equations help students avoid the sign error. These formulas are for compound interest or discount.

11) $PV = A(t)v^t = A(t)(1-d)^t = A(t)(1+i)^{-t}$. So PV with interest uses a negative exponent while PV with discount uses a positive exponent.

12) $A(t) = A(0)(1+i)^t = A(0)v^{-t} = A(0)(1-d)^{-t}$. So AV with interest uses a positive exponent while AV with discount uses a negative exponent.

The *force of interest* is the slope of the accumulation function divided by the amount in the fund at time t:

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{A'(t)}{A(t)}.$$

13) $a(t) = e^{\int_0^t \delta_x dx}$.

14) If $\delta_t = \dfrac{1}{k + ct}$, then $\displaystyle\int_0^t \frac{1}{k + cx} dx = \frac{1}{c} \ln(k + cx)|_0^t$ since $\dfrac{d}{dx}\dfrac{1}{c}\ln(k + cx) = \dfrac{1}{c}\dfrac{1}{k+cx} c = \dfrac{1}{k+cx} = \delta_x$, the integrand.

15) If $\delta_t = \dfrac{c}{k+ct}$, then $\displaystyle\int_0^t \frac{c}{k+cx} dx = \ln(k+cx)|_0^t$ since $\dfrac{d}{dx}\ln(k+cx) = \dfrac{1}{k+cx} c = \dfrac{c}{k+cx} = \delta_x$, the integrand. Note the integrand is c times the integrand in 14).

16) For compound interest, $\delta_t \equiv \delta$ is a constant. So $\displaystyle\int_0^t \delta dx = \delta t$. Hence $a(t) = (1+i)^t = e^{\delta t}$, and $e^\delta = 1 + i$, or $\delta = \ln(1+i)$.

17) It can be shown that $\lim_{n \to \infty} i^{(m)} = \ln(1+i) = \delta$. So δ is the nominal rate of interest compounded continuously.

18) $v = e^{-\delta} = 1 - d$.

Recall that $\dfrac{d}{dt}\ln(f(t)) = \dfrac{f'(t)}{f(t)}$.

19) Suppose you are given $a(t)$, $A(t)$ or δ_t with a variable force of interest. Let $AV = A(t_1, t_2)$ be the accumulated value of \$1 invested at time t_1 and accumulated to time t_2 for $t_2 > t_1$. Let $PV = PV(t_1, t_2)$ be the present value at time t_1 of \$1 due at time t_2 for $t_2 > t_1$. Since the PV at time 0 of \$1 at t_1 is $1/a(t_1)$, $AV = A(t_1, t_2) = \dfrac{a(t_2)}{a(t_1)} = \dfrac{A(t_2)}{A(t_1)} = e^{\int_{t_1}^{t_2} \delta_x dx}$ for $t_2 > t_1$, and $PV = PV(t_1, t_2) = \dfrac{a(t_1)}{a(t_2)} = \dfrac{A(t_1)}{A(t_2)} = e^{-\int_{t_1}^{t_2} \delta_x dx}$ for $t_2 > t_1$.

20) Suppose you are given $a(t)$, $A(t)$ or δ_t with a variable force of interest. Let $A(t_1, t_2)$ be the accumulated value of \$K invested at time t_1 and accumulated to time t_2 for $t_2 > t_1$. Let $PV(t_1, t_2)$ be the present value at time t_1 of \$K due at time t_2 for $t_2 > t_1$. Then $AV = A(t_1, t_2) = \dfrac{Ka(t_2)}{a(t_1)} = \dfrac{KA(t_2)}{A(t_1)} = Ke^{\int_{t_1}^{t_2} \delta_x dx}$ for $t_2 > t_1$, and $PV = PV(t_1, t_2) = \dfrac{Ka(t_1)}{a(t_2)} = \dfrac{KA(t_1)}{A(t_2)} = Ke^{-\int_{t_1}^{t_2} \delta_x dx}$ for $t_2 > t_1$.

21) **An exception to 19) and 20) is simple interest**, the accumulated value of \$K invested at time t_1 and accumulated to time t_2 for $t_2 > t_1$ is $AV(t_1, t_2) = Ka(t_2 - t_1) = K[1 + i(t_2 - t_1)]$ and the present value at time t_1 of \$K due at time t_2 for $t_2 > t_1$ $PV(t_1, t_2) = K[1 + i(t_2 - t_1)]^{-1}$ for $t_2 > t_1$

22) For compound interest, 19) becomes $AV(t_1, t_2) = e^{\int_{t_1}^{t_2} \delta dx} = e^{\delta(t_2 - t_1)} = (1 + i)^{t_2 - t_1} = a(t_2 - t_1)$.

23) So for compound interest and simple interest, the accumulated value of \$K invested at time t_1 and accumulated to time t_2 for $t_2 > t_1$ is $AV = AV(t_1, t_2) = Ka(t_2 - t_1)$ and the present value at time t_1 of \$K due at time t_2 for $t_2 > t_1$ is $PV = PV(t_1, t_2) = K[a(t_2 - t_1)]^{-1}$ for $t_2 > t_1$. Usually want the PV for $t_1 = 0$.

A *time diagram* shows the time periods. It is useful to put deposits and withdrawals on opposite sides of the line. Solve *problems of interest* by setting up *equations of value* as of a *common comparison date.* Write the time period (eg years, 6 months, 3 months) to the left of the time diagram.

24) The *average annual return per year* of an investment made over a t year period is the equivalent annual compound rate of interest that results in $a(t)$ or $A(t)$. So solve for i in the equation $K(1 + i)^t \stackrel{set}{=} Ka(t) = A(t)$ or $(1 + i)^t \stackrel{set}{=} a(t)$.

25) *Equivalent rates:* Two rates are equivalent if both rates produce the same results over the same period of time t in that $A_1(t) = A_2(t)$ or $PV_{1t} = PV_{2t}$.

26) *Equivalent rates for compound interest or constant rates:* the equations in 25) hold for all $t > 0$. So if the problem asks you to find a rate that is equivalent to another rate, usually use $t = 1$ year in writing equations.

27) *Equivalent rates for the variable rate case:* the equations in 25) need the period t to be specified to find equivalent rates. 24) is a special case if $a(t)$ is for a variable rate.

28) The amount of time t in years it takes for money to double at a given rate of compound interest is the solution to $(1+i)^t = 2$ or $t = \dfrac{\ln(2)}{\ln(1 + i)} \approx \dfrac{0.70}{i} = \dfrac{70}{i \text{ in percent}}$.

29) Consider the PV of 1 due at time t. Then $1 = A(t) = PV a(t)$, so $PV = 1/a(t)$. The PV of K due at time t is $K/a(t)$. Note that if $a(t) = (1+i_1)(1+i_2)\cdots(1+i_t)$, then the PV of 1 due at time t is $PV = \frac{1}{1+i_1}\frac{1}{1+i_2}\cdots\frac{1}{1+i_t} = v_{i_1}v_{i_2}\cdots v_{i_t}$. See HW2 4.

30) If the annual inflation rate is r, then on average, \$(1+r) at the end of the year buys what \$1 buys at the beginning of the year.

31) With annual interest rate i and annual inflation rate r, the real rate of interest for the year is $i_{real} = \frac{i-r}{1+r}$ =(value of amount of return in year end dollars)/(value of amount invested in year end dollars).

Chapter 2: An **annuity** is a series of periodic payments.

32) $\sum_{j=0}^{k} x^k = 1 + x + x^2 + \cdots + x^k = \frac{1-x^{k+1}}{1-x} = \frac{x^{k+1}-1}{x-1}$.

33) $S = \sum_{j=0}^{n-1} ar^j = a\sum_{j=0}^{n-1} r^j = a + ar + ar^2 + \cdots + ar^{n-1} = a\,\frac{1-r^n}{1-r}$. Note that a is the 1st term, r is the ratio of successive terms, and n is the number of terms. Also $S = \frac{a - ar^n}{1-r}$ = (1st term − "$(n+1)$th" term)/$(1-r)$ = (1st term) $\frac{1-(\text{ratio})^{\#\text{ of terms}}}{1-\text{ratio}}$.

34) An **annuity-immediate** makes a payment of \$K at the end of the year for n years. **Note that the 1st payment is not immediate.**

35) Consider an annuity-immediate where a payment of \$1 is made at the end of each year for n years so $K = 1$. Then the present value of this annuity is

$a_{\overline{n}|} = v + v^2 + v^3 + \cdots + v^n = \sum_{j=1}^{n} v^j = \frac{v(1-v^n)}{1-v} = \frac{1-v^n}{i}$ since $1 - v = d = iv$. When i is not understood, use $a_{\overline{n}|i} = a_{\overline{n}|}$. Note that $a_{\overline{n}|}$ is read "a angle n" and $a_{\overline{n}|i}$ is read "a angle n at i." **Warning**: $a(t)$ is for AV, but $a_{\overline{n}|}$ is for the PV of an annuity.

36) Consider an annuity-immediate where a payment of \$1 is made at the end of each year for n years so $K = 1$. Then the accumulated value of this annuity is

$s_{\overline{n}|} = s_{\overline{n}|i} = (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1 = \sum_{j=0}^{n-1}(1+i)^j = \frac{(1+i)^n - 1}{i}$.

37) An **annuity-due** makes a payment of \$K at the beginning of the year for n years. **Note that the 1st payment is immediate.**

38) Consider an annuity-due where a payment of \$1 is made at the beginning of each year for n years so $K = 1$. Then the present value of this annuity is

$\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{n}|i} = 1 + v + v^2 + v^3 + \cdots + v^{n-1} = \sum_{j=0}^{n-1} v^j = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$.

Note that $\ddot{a}_{\overline{n}|}$ is read "a double dot angle n."

39) Consider an annuity-due where a payment of \$1 is made at the beginning of each year for n years so $K = 1$. Then the accumulated value of this annuity is

$\ddot{S}_{\overline{n}|} = \ddot{S}_{\overline{n}|i} = (1+i)^n + (1+i)^{n-1} + \cdots + (1+i) = \sum_{j=1}^{n}(1+i)^j = \frac{(1+i)^n - 1}{d} = \frac{(1+i)^n - 1}{iv}$.

40) As a mnemonic for the AV and PV, note that the annuity-**i**mmediate has an **i** in the denominator while the annuity-**d**ue has a **d** in the denominator and the double dots.