Exam 3 is Th. April 5. **You are allowed 10 sheets of notes and a calculator.**

90)-91) may be useful for Exam 3.

103) Consider a time diagram labeled 0, 1, ..., $n$. Put withdrawals on one side and deposits on the other. Place the time period to the left or right of the plot.

If the balance $X$ at time $n$ is wanted, put $X$ at $n$ on the side of the withdrawals since $X$ is the amount you could withdraw at time $n$.

The interest rate $j$ is for the time period. Interest $i^{(m)}$ compounded $m$thly (annually $m = 1$, biannually $m = 2$, quarterly $m = 4$, monthly $m = 12$, daily $m = 365$, weekly $m = 52$, every 2 years $m = 1/2$) is a nominal annual rate, not the actual annual rate $i$, and $i^{(m)} < i$ for $m \geq 2$. Then $j = \frac{i^{(m)}}{m}$ is the period rate, and $\left(1 + \frac{i^{(m)}}{m}\right)^m = 1 + i$.

If $j = \frac{i^{(m_1)}}{m_1}$, but $i^{(m_2)}$ is given, then $(1+j)^{m_1} = \left(1 + \frac{i^{(m_2)}}{m_2}\right)^{m_2}$, so

$j = \left(1 + \frac{i^{(m_2)}}{m_2}\right)^{m_2/m_1} - 1.$

Note that $n = m(\#$ of years).

104) Bond price between coupon dates: Let the actual price paid for the bond = price-plus-accrued of the bond be $P_{t+k} = P_t(1+j)^k = v_j^{1-k}(P_{t+1} + Fr)$ where $P_t$ is the price just after the last coupon was paid at time $t$ and $k$ =(number of days since last coupon was paid)/(number of days in the coupon period). Usually $k$ = (number of months since last coupon)/6. Note that $0 \leq k \leq 1$.

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                bond price at t+1 is just before the coupon
P_t                    P_{t+k}            P_{t+1} + Fr   bond price

-------------------------------------------------
|                      |                  |
t                      t+k                t+1
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Just before the coupon is paid at time $t+1$ the AV of the bond is $P_{t+1} + Fr$. If the bond is bought at time $t+k$, then the buyer gets the coupon $Fr$ paid at time $t+1$. Hence if the interest is $j$, the bond price $P_{t+k}$ is the AV of $P_t$ at time $t$ accumulated to time $t+k$ or $P_{t+k}$ is the PV at time $t+k$ of the value $P_{t+1} + Fr$ of the bond at time $t+1$ just before the coupon is paid. $P_{t+k}$ is approximately the linear interpolation between $P_t$ and $P_{t+1} + Fr$.

Note that $P_0 = P = Fra_{\overline{n}|j} + Cv_j^n$, $P_1 = Fra_{\overline{n-1}|j} + Cv_j^{n-1}$ and $P_t = Fra_{\overline{n-t}|j} + Cv_j^{n-t}$ since $n-t$ is the number of coupons remaining to be paid, where $0 \leq t \leq n$ is an integer.

105) The market price quoted in the financial press is $MP_{t+k} = P_t(1+j)^k - Frk = P_{t+k} - Frk$. This price is smoother and is almost the linear interpolation between $P_t$ and $P_{t+1}$.

106) Like loans, bonds can be amortized. Bond terminology used in place of the loan terminology: a) "book value" is used for "outstanding balance", b) for a bond redeemed at a premium, "writing down" is used instead of "principal repaid", c) for a bond redeemed at a discount, "writing up" is used instead of "principal repaid", d) "coupon" is used instead of "payment amount".

107) Let the book value $BV_t = P_t$. The interest earned is $I_t = jBV_{t-1}$. The principal adjustment is $PR_t = Fr - I_t$, is used for a bond bought at a premium (then $PR_t \geq 0$), but accumulation of discount $AD_t = -PR_t = I_t - Fr$ is used for a bond bought at a premium (then $AD_t \geq 0$). Finally, $BV_t = BV_{t-1} - PR_t = BV_{t-1} + AD_t = BV_{t-1}(1+j) - Fr$.

108) The amortization schedule for the bond is as follows with $PR_t = Fr - I_t$ and $BV_t = BV_{t-1} - PR_t$ used for bonds bought at a premium where the coupon payments reduce book value from $P$ to the redemption value $C$, while $AD_t = I_t - Fr$ and $BV_t = BV_{t-1} + AD_t$ are used for bonds bought at a discount where the coupon payments increase the book value from $P$ to the redemption value $C$.

| duration<br>t | interest<br>payment<br>$Fr$ | interest<br>earned<br>$I_t = (j)BV_{t-1}$ | principal<br>adjustment<br>$PR_t = Fr - I_t$<br>$AD_t = I_t - FR$ | book value<br>$BV_t = BV_{t-1}(1+j) - Fr$<br>$BV_t = BV_{t-1} - PR_t = P_t$<br>$BV_t = BV_{t-1} + AD_t$ |
|---|---|---|---|---|
| 0 | − | − | − | $BV_0 = P = P_0$ |
| 1 | $Fr$ | $I_1 = (j)BV_0$ | $PR_1 = Fr - I_1$ | $BV_1 = BV_0 - PR_1$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| t | $Fr$ | $I_t = (j)BV_{t-1}$ | $PR_t = Fr - I_t$ | $BV_t = BV_{t-1} - PR_t$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| n | $Fr$ | $I_n = (j)BV_{n-1}$ | $PR_n = Fr - I_n$ | $BV_n = BV_{n-1} - PR_n = C$ |
| total | $nFr$ | $\sum_{j=1}^n I_j$ | | |

109) A callable bond is one for which the issuer has the right to redeem the bond at any of several time points: the earliest is the call date $t_1$ and the latest is the usual maturity date $t_2 = n$. General principal: to ensure that the bond earns a specified minimum yield rate $j$, compute the lowest possible price for all possible redemption dates using $j$. This date is the worst date. Two special cases follow. i) The PEW principal says that for a bond selling at a premium ($P > C$ and $g > j$), the earliest redemption date (call date) is the worst. Then $P = Fra_{\overline{t_1}|j} + Cv_j^{t_1}$. ii) The DLW principal says that for a bond selling at a discount ($P < C$ and $g < j$), the latest redemption date (maturity date) is the worst. Then $P = Fra_{\overline{t_2}|j} + Cv_j^{t_2}$. Note that if $t_1 \leq t \leq t_2 = n$ are the possible redemption rates, then the price at time $t$ is $P_t = C + (Cg - Cj)a_{\overline{t}|j}$ which is lowest for $t = t_1$ if $g > j$ so the bond is sold at a premium, and lowest for $t = t_2 = n$ if $g < j$ so the bond is sold at a discount. Recall from 100) that $Fr = Cg$.

**Ch. 5** 110) The internal rate of return (IRR) or yield rate $i$ for a transaction is the interest rate $i$ at which the value of all cashflows out (disbursements or liabilities) $B_0, B_1, ..., B_n$ is equal to all cashflows in (assets) $A_0, A_1, ..., A_n$ received at times $t_0 = 0 < t_1 < \cdots < t_n$ where $A_k \geq 0$ and $B_k \geq 0$. If $C_k = A_k - B_k$ then $i$ satisfies $NPV = \sum_{k=0}^{n} C_k v_i^{t_k} = 0$ where NPV = net present value of the cashflows. The equation of value for PV(0) is $A_0 + A_1 v^{t_1} + \cdots A_n v^{t_n} = B_0 + B_1 v^{t_1} + \cdots B_n v^{t_n}$. The equation of value for AV(n) is $\sum_{k=0}^{n} C_k (1+i)^{t_n - t_k} = 0$.

111) Sometimes the IRR $i$ is unique if $i > -1 = -100\%$, but often there is no real solution $i$ or there are several real solutions to the polynomial $NPV(j) = \sum_{k=0}^n C_k v_j^{t_k} = 0$.

112) The quadratic formula for the roots of $ax^2 + bx + c = 0$ is $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

113) Suppose there are $J$ transactions each with a cashflow vector $\boldsymbol{c}_j = (C_{0j}, C_{1j}, ..., C_{nj})$ for $j = 1, ..., J$. For a fixed interest rate $i$ (called the cost of capital or interest preference rate), the cashflow vector $\boldsymbol{c}_m$ that maximizes the net present value $P_i(\boldsymbol{c}_j) = \sum_{k=0}^{n} C_{kj} v_i^{t_k}$ is preferable.

114) A positive NPV indicates that the investment will be profitable while a negative NPV indicates that the investment will not be profitable. The IRR is the interest rate for which the NPV = 0.

115) The dollar-weighted rate of return for a 1 year period is like an IRR using simple interest. Let the balance at the start of the year be $A$. Let the balance at the end of the year be $B$. For $0 < t_1 < \cdots < t_n < 1$, let $C_k$ be the net cashflow at time $t_k$. Let $C = \sum_{k=1}^{n} C_K$. Then the net amount of interest earned by the fund during the year is $I = B - A - C$ and the dollar-weighted rate of return earned by the fund for the year is $i_D = \dfrac{I}{A + \sum_{k=1}^{n} C_k(1 - t_k)}$. Take a month to be 1/12th of a year. Recall that simple interest has $a(t) = 1 + it$, so $C_k$ at time $t_k < 1$ accumulates to $C_k[1 + i(1 - t_k)]$ at time $t = 1$ using simple interest.

116) As an approximation to 115), assume all $C_k$ are uniformly spread through the year. This assumption is the same as all $t_k \equiv 1/2$ for $k = 1, ..., n$. Then $i_D \approx \dfrac{I}{A + \frac{1}{2}C} = \dfrac{2I}{A + B - I}$. Note $C_k > 0$ for a deposit and $C_k < 0$ for a withdrawal.

117) If compound interest was used in 115), then $A(1 + i) + \sum_{k=1}^{n} (1 + i)^{1-t_k} = B$. The difference between the $i$ in 115) using simple interest and the $i$ using compound interest for periods $\leq 1$ year are usually not large.

118) Let the balance at the start of the year be $A$. Let the balance at the end of the year be $B$. For $0 < t_1 < \cdots < t_n < 1$, let $C_k$ be the net cashflow at time $t_k$. Let $F_k$ be the value of the fund just before the net cashflow deposit of $C_k$ at time $t_k$. Then the time-weighted return earned by the fund at the end of the year is

$$i_T = \left[ \left(\frac{F_1}{A}\right) \left(\frac{F_2}{F_1 + C_1}\right) \left(\frac{F_3}{F_2 + C_2}\right) \cdots \left(\frac{F_k}{F_{k-1} + C_{k-1}}\right) \left(\frac{B}{F_k + C_k}\right) \right] - 1.$$

So $(1 + i_T)$ is the term in the brackets.

119) To compute $i_T$ in 118), make a table. Often problems use months and time 0 $\approx 1/2$ (Jan. 2 of current year), and $1 \approx 1/1$ (Jan. 1 of following year). Note that for a deposit $C_k > 0$ and for a withdrawal $C_k < 0$.

| | 0 | $t_1$ | $t_2$ | $\cdots$ | $t_k$ | 1 |
|---|---|---|---|---|---|---|
| amount before net deposit = $F_k$ | | $F_1$ | $F_2$ | $\cdots$ | $F_k$ | $B$ |
| net deposit $C_k$ | | $C_1$ | $C_2$ | $\cdots$ | $C_k$ | |
| amount after net deposit = $F_k + C_k$ | $A$ | $F_1 + C_1$ | $F_2 + C_2$ | $\cdots$ | $F_k + C_k$ | |

120) Sometimes values such as deposits are given both at the end of the month, eg Jan. 31, or the beginning of the month, eg Feb. 1. For the time diagram, the end of the time period and the beginning of the next time period should be considered to be the

same time. Do not put Feb. 1 one time period after Jan. 31 for monthly data.

121) In an investment group, the members of the group pool investments and earnings. The earnings are credited to the members in terms of their (in proportion to the amount) of the member's investment. The portfolio method gives the same investment rate to each member. The investment yield method (IYM) gives rates that are creditied for each year, but after $m$ years the the funds are often pooled together and the porfolio method is used.

122) The symbol $i_t^y$ for IYM stands for the interest rate credited on an investment made on Jan. 1 of year $y$ in the $t$th year of the investment. Then $i_1^y$ is the *new money rate*. So $i_1^{2005}, i_2^{2005}, ...$ stand for interest rates credited at the end of 2005, 2006, ..., while $i_1^{2006}, i_2^{2006}, ...$ stand for interest rates credited at the end of 2006, 2007, .... The symbol $i^y$ is the portfolio interest rate credited for year $y$. So $i^{y+m}$ is a portfolio interest rate.

| $y$ | investment $i_1^y$ | year $i_2^y$ | rates % $i_3^y$ | $i_4^y$ | $i_5^y$ | portfolio rate $i^{y+5}$ % | year for portfolio rate |
|---|---|---|---|---|---|---|---|
| 1992 | | | | | Q | 8.35 | 1997 |
| 1993 | | | | | Q | 8.6 | 1998 |
| 1994 | | | use | for | Q | 8.85 | 1999 |
| 1995 | | | | | | 9.1 | 2000 |
| 1996 | P | P | P | | | 9.35 | 2001 |
| 1997 | R 9.5 | 9.5 | 9.6 | 9.7 | 9.7 | | |
| 1998 | R 10 | 10 | 9.9 | 9.8 | | | |
| 1999 | R 10 | 9.8 | 9.7 | | | | |
| 2000 | 9.5 | 9.5 | | | | | |

123) Problems can be illustrated by the above table and this example. Let Jo invest 1000 on Jan. 1, 1997. Let the following be the AV of the 1000 on Jan. 1, 2000:
$P$ under the investment year method
$Q$ under the portfolio method
$R$ where the balance is withdrawn each year and reinvested under the new money rate.

Then Jan. 1, 2000 "=" Dec. 31, 1999 and interest is earned for 3 years: 1997, 1998, and 1999. For this example, $m = 5$.

$P = 1000(1 + i_1^{1997})(1 + i_2^{1997})(1 + i_3^{1997}) = 1000(1.095)(1.095)(1.096) = 1314.13.$

$Q = 1000(1 + i^{1997})(1 + i^{1998})(1 + i^{1999}) = 1000(1.0835)(1.086)(1.0885) = 1280.82.$

$R = 1000(1 + i_1^{1997})(1 + i_1^{1998})(1 + i_1^{1999}) = 1000(1.095)(1.1)(1.1) = 1324.95.$

Also if $m = 5$ and $y = 1997$ then using IYM for 7 years gives
$AV = 1000(1 + i_1^{1997})(1 + i_2^{1997})(1 + i_3^{1997})(1 + i_4^{1997})(1 + i_5^{1997})(1 + i^{2002})(1 + i^{2003}).$

124) Suppose deposits and withdrawals are made continuously and let $F(t)$ be the amount in the fund at time $t$. Let $I = I_{t_1 t_2}$ be the interest "earned" on $F(t_1)$ and on the withdrawals and deposits made from time $t_1$ and $t_2$. Then the IRR or yield rate from time $t_1$ to $t_2$ is $i \approx \dfrac{2I}{F(t_1) + F(t_2) - I}$.

**chapter 6**

125) Stocks and bonds are basic investments. Stock pays dividends, usually quarterly, at the company's discretion. If the corporation issues a bond, then the bond debt must be paid before the preferred stock dividend, which must be paid before common stock dividends. So a corporation bond is more secure (has less risk) than preferred stock which is more secure than common stock. Look at the past 5 years for dividends and

P/E ratio = price per share/earnings per share. Roughly 40% of stock earnings come from dividends and 60% from stock market going up about 10% a year over time.

126) A zero coupon bond is a bond that has a single payment made at the time of the maturity with no other payments. Let P = PV = price = principal = X. Let $Y = A(t) = F = AV$. Then 6) on the exam 1 review can be used to find $P, F, i$ or $t$ given the other 3 quantities.

| years to maturity | spot rate $s_j$ = % yield rate on zero coupon bond |
|---|---|
| 1 | $s_1 = 5.00$ |
| 2 | $s_2 = 5.35$ |
| 3 | $s_3 = 5.65$ |
| 4 | $s_4 = 5.90$ |
| 5 | $s_5 = 6.06$ |
| $\vdots$ | $\vdots$ |
| 10 | $s_{10} = 6.50$ |

127) The above table shows the *term structure of the interest rates*: how yield rates vary by term of investment. The graph of the table data is called the yield curve. The term structure is *normal* if longer term investments offer higher interest rates than shorter term investments. The term structure is *inverted* if longer term investments offer lower interest rates than shorter term investments. The term structure is *flat* if the yield curve is flat so the interest rates are nearly constant (do not depend on the term). The tangent line to the yield curve will be positive for periods of normal term structure, negative for periods of inverted term structure and approximately 0 for periods of flat term structure.

128) The *spot rate* is the yield rate for an investment that makes a lump sum payment to the investor at a known date. Zero coupon bonds are such an investment, so the table above 127) gives spot rates. If the investment makes the lump sum payment at the $j$th time period, denote the spot rate by $s_j$. So for the table above 127), $s_1 = 0.05, ..., s_{10} = 0.065$.

```
     C1   C2     ...     Cn

 -----------------------------
 |    |    |     ...     |
 0    t1   t2            tn     time
 PV   s1   s2            sn     spot rate
```

129) Suppose you purchase zero coupon bonds with $n$ maturity dates that pay $C_1, ..., C_n$ at the maturity dates $t_1, ..., t_n$ with spot rates $s_1, ..., s_n$. See the above time diagram. Then the present value of the investment is $PV = \sum_{j=1}^{n} C_j v_{s_j}^{t_j} = C_1 v_{s_1}^{t_1} + C_2 v_{s_2}^{t_2} + \cdots + C_n v_{s_n}^{t_n}$, which is the sum of the $n$ present values. Often $C_j \equiv C$ and $t_j = j$ for $j = 1, ..., n$. Note that $v_{s_j} = (1 + s_j)^{-1}$.

130) Suppose $PV = \sum_{j=1}^{n} C_j v_{s_j}^{j}$ where the PV and the $s_j$ are given except for $s_k$. Then $PV = C_1 v_{s_1} + C_2 v_{s_2}^2 + \cdots + C_k v_{s_k}^k + \cdots + C_n v_{s_n}^{t_n}$, or $C_k v_{s_k}^k = C_k(1 + s_k)^{-k} = PV - C_1 v_{s_1} - \cdots - C_{k-1} v_{s_{k-1}}^{k-1} - C_{k+1} v_{s_{k+1}}^{k+1} - \cdots - C_n v_{s_n}^n = A$. So $s_k = \left(\frac{A}{C_k}\right)^{\frac{-1}{k}} - 1$.

```
     Fr   Fr    ... Fr    Fr+C
 -----------------------------
 |   |    |     ... |     |
 0   1    2         2n-1  2n      six months
 PV  s1   s2        s2n-1 s2n     spot rate
```

131) A stripped security is a stripped treasury bond or note where each coupon is sold as a zero coupon security as is the residual = maturity payment = C. If an $F$ $n$ year bond has coupon rate $r$ with semiannual coupons and redemption price $C$, then as a STRIP the bond has $2n-1$ coupons of $Fr$ paid at time in years $0, 0.5, 1, 1.5, ..., n-1, n-0.5$ and one payment of $Fr+C$ at time $n$. Typically the term structure table gives the maturity date in years and the nominal bond yield rate compounded semiannually: the two middle rows in the below table. Then the time diagram is as above. Then the price of the entire STRIP (of the $2n$ coupons and the derivative) is $PV = Fr[v_{s_1}^1 + v_{s2}^2 + \cdots + v_{s_{2n-1}}^{(2n-1)}] + (Fr+C)v_{s_{2n}}^{2n} = Fr[(1+s_1)^{-1} + (1+s_2)^{-2} + \cdots + (1+s_{2n-1})^{-(2n-1)}] + (Fr+C)(1+s_{2n})^{-2n}$. Recall that $C = F$ if the bond is redeemed at par, and assume $F = C$ unless told otherwise.

| time in 6 months | maturity date in years | nominal bond yield rate compunded semiannually | spot rate $s_j = i_j^{(2)}/2$ |
|---|---|---|---|
| 1 | 0.5 | $i_1^{(2)}$ | $s_1$ |
| 2 | 1 | $i_2^{(2)}$ | $s_2$ |
| 3 | 1.5 | $i_3^{(2)}$ | $s_3$ |
| 4 | 2 | $i_4^{(2)}$ | $s_4$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 2n | n | $i_{2n}^{(2)}$ | $s_{2n}$ |

132) The theoretical price of the STRIP is the price of the bond as given in 100).

133) A forward rate is an interest rate that will be earned on an investment made at a future point in time. Let $f_t = i(t, t+1) = i_{t,t+1}$ be the $t$-year forward rate for year $t+1$ starting at time $t$. Define $f_0 = i(0,1) = i_{0,1} = s_1$.

134) Let $a_Z(n) = AV$ of 1 invested now in an $n$ year zero coupon bond. Given spot rates, $s_1, ..., s_n$, then $a_Z(n) = (1+s_n)^n = \prod_{j=0}^{n-1}(1+f_j) = (1+f_0)(1+f_1)\cdots(1+f_{n-1}) = \prod_{k=1}^{n}(1+i(k-1,k)) = (1+i(0,1))(1+i(1,2))\cdots(1+i(n-1,n))$.

135) **Know**: $1 + i(n, n+1) = 1 + f_n = \dfrac{(1+s_{n+1})^{n+1}}{(1+s_n)^n} = \dfrac{a_Z(n+1)}{a_Z(n)}$.

136) If a zero coupon bond matures for $K$ at time $n$ then the AV $= K$ and the price of the bond $= PV = K(1+s_n)^{-n} = K\prod_{j=0}^{n-1}(1+f_j)^{-1} = K(1+f_0)^{-1}(1+f_1)^{-1}\cdots(1+f_{n-1})^{-1}$.