Exam 4 is Th. April 24. **You are allowed 13 sheets of notes and a calculator.**

ch. 7: 137) Unless told otherwise, "duration" refers to Macaulay duration. The duration of a single cashflow is the time remaining until maturity, eg n. The greater the duration, the more sensitive the PV is to changes in the interest rate. The units of a duration are the length of the time period, usually years.

138) Let $P = P(i) = PV$ = price of the investment as a function of the interest rate i. The **Macaulay duration** = MacD = $D_{Mac} = D = \frac{-P'(i)}{P(i)}(1+i) = \frac{\frac{-d}{di}P(i)}{P(i)}(1+i)$.

139) The **modified duration** = ModD = $D_{Mod} = DM = \frac{-P'(i)}{P(i)} = \frac{\frac{-d}{di}P(i)}{P(i)}$ = price sensitivity with respect to interest. The change in price = the change in PV = $P(i+h) - P(i) \approx -h\ P(i)\ D_{Mod}$ for a small change h in interest.

140) Suppose an investment is bought at time 0 and has cashflows C_t at times $t = 1, ..., n$. Then $PV = P(i) = \sum_{t=1}^n C_t v_i^t = \sum_{t=1}^n C_t(1+i)^{-t} = \sum_{t=1}^n C_t e^{-\delta t}$, and the Macaulay duration $D = D_{Mac} = \frac{\sum_{t=1}^n tC_t v_i^t}{\sum_{t=1}^n C_t v_i^t} = \frac{\sum_{t=1}^n tC_t(1+i)^{-t}}{\sum_{t=1}^n C_t(1+i)^{-t}} = \frac{\sum_{t=1}^n tC_t e^{-\delta t}}{\sum_{t=1}^n C_t e^{-\delta t}}$. Note that the numerator summand = (t)(denominator summand). Given the present value is a sum, put the sum in the denominator and put a sum with summand = (t) (denominator summand) in the numerator to find the duration. Note that t is the exponent on v_i and $-t$ is the exponent on $(1+i)$ in the denominator summand.

141) $D_{Mac} = D = \frac{\frac{-d}{d\delta}P(i)}{P(i)}$ = price sensitivity with respect to force of interest.

142) $D = D_{Mac} = D_{Mod}(1+i) = D_{Mod}\ v_i^{-1}$, and $D_{Mod} = \frac{D_{Mac}}{1+i} = D_{Mac}\ v_i$.

143) Note that $D_{Mac} = \sum_{t=1}^n w_t C_t$ where $w_t = \frac{C_t(1+i)^{-t}}{P(i)}$ and $\sum_{t=1}^n w_t = 1$. Hence the Macaulay duration is a weighted average of the cashflow times t with weights w_t.

144) Suppose $C_t \equiv K$ for $t = 1, ..., n$. Often the duration $D = D_{Mac}$ can be computed by recognizing that term in the sums are PVs of annuities. For example, the PV of an annuity immediate with $K = 1$ is $a_{\overline{n}|} = \sum_{t=1}^n v_i^t$. See 35). The PV of an increasing annuity immediate with $K = 1$ is $(Ia)_{\overline{n}|} = \sum_{t=1}^n t v_i^t$. See 71). Using a time diagram to get the PV can be useful.

a) The duration of an annuity immediate with $PV = Ka_{\overline{n}|}$ is $D = D_{Mac} = \frac{K\sum_{t=1}^m t v_i^t}{K\sum_{t=1}^m v_i^t} = \frac{(Ia)_{\overline{n}|i}}{a_{\overline{n}|i}}$.

b) Consider a bond that matures in n years with redemption value C that pays annual coupons of Fr at times 1, ..., n with yield rate $j = i$. Then $PV = P(i) = Fr\sum_{t=1}^n v^t + Cv^n$. Hence the duration $D = D_{Mac} = \frac{Fr\sum_{t=1}^n t\ v^t + C\ n\ v^n}{Fr\sum_{t=1}^n v^t + Cv^n} = \frac{Fr(Ia)_{\overline{n}|i} + C\ n\ v_i^n}{Fra_{\overline{n}|i} + Cv^n}$.

c) An $n-$year bond with annual coupons $K = Fr$ that sells at its par value has $P = P(i) = F = C$ and $r = g = j = i$. Then the duration is $D = \ddot{a}_{\overline{n}|i}$.

145) Suppose there is a portfolio of m investments with $PV = P_k(i) = X_k$ for $k = 1, ..., m$. Denote the cashflows or the kth investment by $C_{1k}, ..., C_{nk}$. Let D_k be the Macaulay duration of the kth investment, and let the PV of the portfolio be $P = P(i) = X = \sum_{k=1}^m X_k = \sum_{k=1}^m P_k(i)$. Let $D = D_{Mac}$ be the Macaulay duration of the portfolio. Then $D = \dfrac{\sum_{k=1}^m D_k P_k(i)}{\sum_{k=1}^m P_k(i)} = \sum_{k=1}^m w_k D_k$ where $w_k = \dfrac{P_k(i)}{P(i)}$ and $\sum_{k=1}^m w_k = 1$. Hence the duration of the portfolio is the weighted average of the durations D_k of the investments with weights w_k.

146) Suppose an investment is bought at time 0 and has cashflows C_t at times $t = 1, ..., n$. Then $PV = P(i) = \sum_{t=1}^n C_t v_i^t = \sum_{t=1}^n C_t(1+i)^{-t} = \sum_{t=1}^n C_t e^{-\delta t}$, and the convexity of the asset is $\frac{\frac{d^2}{di^2}P(i)}{P(i)}$. Hence the convexity $= \dfrac{\sum_{t=1}^n t(t+1)C_t v_i^{t+2}}{\sum_{t=1}^n C_t v_i^t}$. Note that the numerator summand $= [t(t+1)v^2]$ (denominator summand).

147) A financial enterprise will have assets $A_t \geq 0$ and liabilities $L_t \geq 0$. Let the PV of the assets = price of the assets be $P_A(i) = \sum_{t=0}^n A_t v_i^t$, and let the PV of the liabilities be $P_L(i) = \sum_{t=0}^n L_t v_i^t$. Want assets to be enough to pay off liabilities, and *immunization* is the process of protecting a financial enterprise from small changes in the interest rate in that $P_A(i) \geq P_L(i)$ for $i \in (i_0 - \epsilon, i_0 + \epsilon)$ where i_0 is the interest rate at time $t = 0$.

148) The net present value of the assets and liabilities is $NPV(i) = g(i) = P_A(i) - P_L(i) = \sum_{t=0}^n C_t v_i^t$ where $C_t = A_t - L_t$. Redington immunization has three sufficient conditions, and there are three equivalents sets of the sufficient conditions.

condition	set I)	set II)	set III)
i)	$P_A(i_0) = P_L(i_0)$	$P_A(i_0) = P_L(i_0)$	$NPV(i_0) = 0$
ii)	duration of assets = duration of liabilities	$P'_A(i_0) = P'_L(i_0)$	$NPV'(i_0) = 0$
iii)	convexity assets > convexity liabilities	$P''_A(i_0) > P''_L(i_0)$	$NPV''(i_0) > 0$

For the first set of sufficient conditions, the convexity and durations are evaluated at $i = i_0$. The Macaulay duration or the modified duration can be used.

For condition i), equality can be replaced by $\geq$, at least for the sets of conditions II) and III), but most texts use equality. Condition i) makes $g(i_0) = NPV(i_0) = 0$. Conditions ii) and iii) insure that $g(i) = NPV(i)$ has a relative min at $i = i_0$. Hence $g(i) \geq g(i_o) = 0$ for i in a small neighborhood of i_0, say for $i \in (i_0 - \epsilon, i_0 + \epsilon)$. Hence $P_A(i) \geq P_L(i)$ for small changes in i from i_0, and Redington immunization protects the financial enterprise from such small changes in interest.

149) A portfolio is *fully immunized* if $\sum_{t=0}^n A_t v_i^t \geq \sum_{t=0}^n L_t v_i^t$ for any $i > 0$. Then the enterprise is protected against any change in the interest rate (provided $i > 0$). There are also 3 sufficient conditions:

i) $P_A(i_0) = P_L(i_0)$

ii) $D_{Mod}(i_0)$ of assets = $D_{Mod}(i_0)$ of liabilities or, equivalently, $P'_A(i_0) = P'_L(i_0)$

iii) There is one asset cash flow before and after any liability cash outflow.

150) Full immunization implies Redington immunization.

151) *Exact matching* (also known as dedication, asset-liability matching, absolute matching and cash flow matching) is a form of immunization. Exact matching has $A_t = L_t$ for all t (eg $t = 0, 1, ..., n$). Often can not predict when the cash inflows and outflows will occur, so exact matching is not possible.

152) The dividend discount model for the theoretical price of a stock is the PV of a perpetuity of dividends. This price is for a stock that will be held many years. The theoretical price from the model is not a very accurate estimate of the actual price. If the stock is bought at time 0 and the dividends are D_t for $t = 1, 2, 3, ...$, then the price $P = PV = \sum_{t=1}^{\infty} D_t v_i^t$. Often dividends are given quarterly, biannually or annually. The interest rate i should be for the time period. If an interest rate is given for a period other than the time period, convert that interest rate into i.

153) Some special cases of 152) are a) $D_t \equiv D$ for $t = 1, 2, ...$ Then
$P = PV = Da_{\overline{\infty}|} = D/i$.
b) The first dividend at time $t = 1$ is D and the dividend increases by a constant ratio $(1 + k)$ for each subsequent period where $k < i$. Then $P = PV =$

$$Dv + Dv^2(1+k) + Dv^3(1+k)^2 + Dv^4(1+k)^3 + \cdots = \sum_{t=1}^{\infty} \frac{D}{1+k}\left(\frac{1+k}{1+i}\right)^t = \frac{D}{i-k}.$$

154) If the dividend D_t has spot rates s_t and $t-$year forward rates f_t, then the price of the stock from the dividend discount model is

$$P = PV = \sum_{t=1}^{\infty} D_t v_{s_t}^t = \sum_{t=1}^{\infty} D_t(1+s_t)^{-t} = \sum_{t=1}^{\infty} D_t[\prod_{j=0}^{t-1}(1+f_j)]^{-1}.$$

155) Be familiar with mutual funds (including index funds), money market funds, and certificates of deposit.

Financial Derivatives including Ch. 9 and $\oint$ 6.4:

156) A *derivative* or *financial derivative* is a financial instrument (contract or agreement) whose value is based on an underlying asset such as a stock. A derivative derives its value from the underlying asset. Derivatives can be thought of as bets on the price of something. Forwards, options, futures and swaps are derivatives. Constructing derivatives from other financial products is known as financial engineering.

157) Uses of derivatives include i) *risk management* and *hedging*: reduce risk, for example due to price changes, by entering into a derivatives contract; ii) pure speculation: want to make a bet, not reduce risk; iii) reduce transaction costs: often the underlying asset is not bought and sold, so the transaction costs for buying and selling the asset are avoided; iv) risk sharing: insurance, for example, is a financial derivative where the vast "lucky majority" lose the premium and the "unlucky few" get money to cover their losses and damages; v) can sometimes defer taxes, et cetera.

158) It is assumed that there is a *risk free rate of interest* (i or r or δ) such as the yield rate on US treasury bills. This risk free rate is typically quoted as a continuously compounded rate = force of interest = δ. It will be assumed that any fraction of an asset, such as a half share of stock, can be bought.

159) Usually need to pay a commission for a purchase or sale: eg \$10 for both buying and selling shares of stock, or a percentage fee such as 0.1% of the value of the purchase or sale. You sell shares to the broker at the **bid price** and buy shares from the broker at the **ask price** ($>$ bid price). The bid and ask price are from the perspective of the "market makers" (brokers) who make it possible to buy and sell the asset (eg shares of stock), almost instantaneously. The *bid ask spread* = ask price $-$ bid price.

160) Suppose the financial derivative is a bet on the future price of an underlying asset. The buyer of the asset at market value (time $t = 0$) has the **long position** and is

betting that the price of the asset will increase. The short seller has the **short position** and is betting that the price of the asset will decrease. If the price of the asset goes down, the holder of the long position will have a loss while the short seller will have a profit (if transaction fees, interest, et cetera are not too high). If the price of the asset goes up, the holder of the long position will have a profit while the holder of the short position will have a loss.

161) A short position is taken by borrowing the asset from the holder of the long position. The asset is immediately sold, but the holder of the short position purchases the asset and returns it to the holder of the long position at the time the short sale is terminated. The act of buying the asset at the time that the short sale is terminated is known as *closing* or *covering* the short position.

162) i) The borrowed assets are sold immediately by the short seller. The proceeds belong to the short seller but are held by the lender (holder of the long position) or by a designated financial institution. Often interest paid on the proceeds is a cost to the short seller if the interest rate is less than that of the risk free interest rate. ii) Additional collateral, known as a *haircut*, may also be required. Interest at a rate known as the *short rebate* for stocks and the *repo rate* for bonds is paid to the short seller. If this rate is less than the risk free interest rate, then the loss of interest income is a cost for the short seller and a source of profit for the holder of the long position. iii) The short seller must pay the lender any dividends paid during the lending period.

163) Ignoring interest gains or losses from a short sale of stock, the cash inflow from selling K shares of borrowed stock $= K(\text{bid price}) -$ commission, while the cash outflow from buying K shares of stock to close the short position $= K(\text{ask price}) +$ commission. If the commission is based on a commission rate that is a small decimal like $0.00c$, then the cash inflow is $K(\text{bid price})(1 - 0.00c)$, while the cash outflow is $K(\text{ask price})(1 + 0.00c)$. Then the cash inflow uses the smaller numbers (bid price and $(1-0.00c)$), while the cash outflow uses the larger numbers (ask price and $(1 + 0.00c)$). See HW11 2.

164) A **forward contract** is an agreement to buy or sell a certain *underlying asset* at a specific future date called the *expiration date* or *delivery date* T for a specific price called the *forward price* or *delivery price* $F = F_{0,T}$. Let time $t = 0$ be the time when the contract is made and let the *spot price* S_0 be the value of the asset at time 0 and let S_T be the value of the asset at time T. The buyer at time T has the long position and the seller at time T has the short position. At time T the long payoff $= S_T - F_{0,T}$ and the short payoff $= F_{0,T} - S_T$. Note that the 2 payoffs always sum to 0. If these payoffs are for one unit and there are K units, then the long payoff $= K(S_T - F_{0,T})$ and short payoff $= K(F_{0,T} - S_T)$. Note that the holder of the short position again makes a profit if the price goes down and a loss if the price goes up. The holder of the long position makes a profit if the price goes up ($S_T > F_{0,T}$) and a loss if the price goes down ($S_T < F_{0,T}$). The holder of the short position sells the asset for $KF_{0,T}$ to the holder of the long position at time T when the market price of the asset is KS_T. The forward contract is called a long forward for the long position and a short forward for the short position.

165) The payoff is the value of the contract to one of the parties at a particular date.

166) In general, the **long position** makes a profit if the price of the underlying asset increases, while the **short position** makes a profit if the price of the underlying asset decreases.

167) For a **call option** or *call*, the *purchaser* (buyer or holder) of the call option has the right (option) to buy or not buy, from the *writer* = seller of the option, the underlying asset at a prespecified time (expiration date) at a specified price K called the *strike price* or *exercise price*. The writer of the option must sell the asset at the specified price K if the purchaser decides to buy the asset, called *exercising* the call option. If the purchaser decides not to exercise the option, then the option expires.
A mnemonic for a call option is COB: a **c**all is an **o**ption to **b**uy the underlying asset for the purchaser of the call.

168) There are two main styles of (call or put) options. The names of the styles have nothing to do where the options are written or traded. Either style can be traded in the US or Europe.
i) The **E**uropean (call or put) option can only be exercised at the **e**xpiration date.
ii) The **A**merican (call or put) option can be exercised **a**nytime during the life of the option.

169) A **payoff** at time t does not depend on cashflows at times other than t. The time $t = T$ is especially important.

170) The *purchased E. call option payoff* = *long call payoff* = $\max(0, S_T - K)$ while the *written E. call option payoff* = *short call payoff* = $-\max(0, S_T - K)$
where S_T is the market price of the asset at time T. Note that the sum of the two payoffs is 0. It is assumed that the call option will be exercised (the asset will be bought) if the spot price of the asset at expiration date $S_T > K$ = exercise price, and not exercised otherwise.

171) Let $C = C(K, T)$ be the call option premium. Let the AV of C at time T be $FV(C)$. If the interest rate is i and the force of interest = rate of interest compounded continuously is r, then $FV(C) = C(1+i)^T = Ce^{rT}$. Sometimes T is the end of one time period (like a year or 6 months), and the interest rate i and force of interest r are given for that time period. Then $FV(C) = C(1+i) = Ce^r$.

172) The *purchased E. call option profit* = *long call profit* = $\max(0, S_T - K) - FV(C)$ while the *written E. call option profit* = *short call profit* = $FV(C) - \max(0, S_T - K)$.
Note that the sum of the two profits is 0.

173) If the premium, payoffs and profits in 170) and 172) are for one unit, and there are J units, then the *purchased E. call option payoff* = *long call payoff* = $J \max(0, S_T - K)$,
written E. call option payoff = *short call payoff*= $-J \max(0, S_T - K)$,
the *purchased E. call option profit* = *long call profit* = $J[\max(0, S_T - K) - FV(C)]$, and the *written E. call option profit* = *short call profit* = $J[FV(C) - \max(0, S_T - K)]$.

174) For the call option, the purchaser has the long position (wants the price to increase from K), and the writer has the short position (wants the price to decrease from K).

175) For a **put option** or *put*, the *purchaser* (buyer or holder) of the put option has the right (option) to sell or not sell, to the *writer* = seller of the option, the underlying asset at a prespecified time (expiration date) at a specified price K called the *strike price* or *exercise price*. The writer of the option must buy the asset at the specified price K if the purchaser decides to sell the asset, called *exercising* the put option. If the purchaser decides not to exercise the option, then the option expires.
A mnemonic for a put option is POS: a **p**ut is an **o**ption to **s**ell the underlying asset for

the purchaser of the put.

176) Let $P = P(K, T)$ be the put option premium. Let the AV of P at time T be $FV(P)$. If the interest rate is i and the force of interest = rate of interest compounded continuously is r, then $FV(P) = P(1+i)^T = Pe^{rT}$. Sometimes T is the end of one time period (like a year or 6 months), and the interest rate i and force of interest r are given for that time period. Then $FV(P) = P(1+i) = Pe^r$.

177) The *purchased E. put option payoff = long put payoff* $= \max(0, K - S_T)$ while the *written E. put option payoff = short put payoff* $= -\max(0, K - S_T)$
where S_T is the market price of the asset at time T. Note that the sum of the two payoffs is 0. It is assumed that the put option will be exercised (the asset will be sold) if the spot price of the asset at expiration date $S_T < K$ = exercise price, and not exercised otherwise.

178) The *purchased E. put option profit = long put profit* $= \max(0, K - S_T) - FV(P)$ while the *written E. put option profit = short put profit* $= FV(P) - \max(0, K - S_T)$. Note that the sum of the two profits is 0.

179) If the premium, payoffs and profits in 177) and 178) are for one unit, and there are J units, then the *purchased E. put option payoff = long put payoff* $= J\max(0, K - S_T)$, *written E. put option payoff = short put payoff* $= -J\max(0, K - S_T)$
the *purchased E. put option profit = long put profit* $= J[\max(0, K - S_T) - FV(P)]$, and the *written E. put option profit = short put profit* $= J[FV(P) - \max(0, K - S_T)]$.

180) For the put option, the purchaser has the short position (wants the price to decrease from K), and the writer has the long position (wants the price to increase from K).

term based on position in option	alternative term	position in underlying asset
long call	purchased call	long
short call	written call	short
long put	purchased put	short
short put	written put	long

181) The owner of the option has the long position in the option since the owner benefits if the price of the option goes up. The writer has the short position in the option since the seller benefits if the price of the option decreases. For a call, the long and short positions in the option are the same as those in the underlying asset. For a put, a long position in the put has a short position in the underlying asset, and a short position in the put has a long position in the underlying asset. The terminology in the leftmost column of the above table refers to the long or short position in the option, not in the underlying asset. Purchased options have the long position in the option while written options have the long position in the option.

182) Given premiums, strike prices, interest rates i, force of interest r = rate of interest rate compounded continuously, be able to find the spot price S_T given the profit or payoff. Be able to find the profit and payoff, given the spot price S_T, using 170), 172), 173), 177), 178) and 179). The premium is the price of the option.

183) The call option is similar to a long forward, but protects the purchaser of the call option from a large price decrease in the underlying asset. The put option is similar to a short forward, but protects the purchaser of the put option from a large price increase in the underlying asset. The cost for this "insurance" is the premium.