Math 401 Exam 3 is Wed. Nov. 19. **You are allowed 10 sheets of notes and a calculator.** The exam covers Q7-10. Know 88) – 95) on exam 2 review.

96) **Formulas are given for unit payment.** For nonunit payment d, multiply the unit payment formula for A by d and the unit formula payment for ${}^2A$ by d^2.

97) Suppose (x) buys insurance and dies at $t \in (k-1, k]$ years from purchase so $K_x = k$ where $k \in \{1, 2, 3, ...\}$. Given v, i or δ and a small table of k and $P(K_x = k)$, be able to find the following quantities for the following 4 discrete life insurance models where a unit payment (eg of \$100000, \$500000 or \$1000000) is made.

i) (Discrete) *whole life insurance* makes unit payment at time $t = k$ with $v_t = v^t, t \geq 0$ and $b_t = 1, t \geq 0$. Then $z_t = b_t v_t = v^t, t \geq 0$. The present value random variable $Z_x = z_{K_x} = v^{K_x}$. Then the actuarial present value APV = EPV = NSP =

$$A_x = E(Z_x) = E(v^{K_x}) = \sum_{k=1}^{\infty} v^k P(K_x = k),$$

$$\text{and} \quad {}^2\mathrm{A_x} = \mathrm{E}[(\mathrm{Z_x})^2] = \mathrm{E}[(\mathrm{v}^{\mathrm{K_x}})^2] = \sum_{\mathrm{k}=1}^{\infty} \mathrm{v}^{2\mathrm{k}} \mathrm{P}(\mathrm{K_x} = \mathrm{k}).$$

ii) (Discrete) *n year term insurance* = (discrete) *n year temporary insurance* makes unit payment at time $t = k$ only if $k \leq n$, otherwise no payment is made. Now
$v_t = v^t, t \geq 0$,

$$b_t = \begin{cases} 1, & t \leq n \\ 0, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t} = \mathrm{b_t v_t} = \begin{cases} v^t, & t \leq n \\ 0, & t > n. \end{cases}$$

The present value random variable

$$Z^1_{x:\overline{n}|} = \begin{cases} v^{K_x}, & K_x \leq n \\ 0, & K_x > n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$$A^1_{x:\overline{n}|} = E(Z^1_{x:\overline{n}|}) = \sum_{k=1}^{n} v^k P(K_x = k),$$

$$\text{and} \quad {}^2\mathrm{A}^1_{\mathrm{x}:\overline{\mathrm{n}}|} = \mathrm{E}[(\mathrm{Z}^1_{\mathrm{x}:\overline{\mathrm{n}}|})^2] = \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{v}^{2\mathrm{k}} \mathrm{P}(\mathrm{K_x} = \mathrm{k}).$$

The 1 above the x means unit benefit is payable after (x) dies if death is not after time n.

iii) (Discrete) *n year deferred insurance* makes unit payment at time $t = k$ only if $k > n$ so $k \geq n+1$, otherwise no payment is made. Now $v_t = v^t, t \geq 0$,

$$b_t = \begin{cases} 0, & t \leq n \\ 1, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t} = \mathrm{b_t v_t} = \begin{cases} 0, & t \leq n \\ v^t, & t > n. \end{cases}$$

The present value random variable

$$_{n|}Z_x = \begin{cases} 0, & K_x \leq n \\ v^{K_x}, & K_x > n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$$ {}_{n}|A_x = E({}_{n}|Z_x) = \sum_{k=n+1}^{\infty} v^k P(K_x = k), $$

$$ \text{and} \quad {}^2\, {}_{n}|A_x = E[({}_{n}|Z_x)^2] = \sum_{k=n+1}^{\infty} v^{2k} P(K_x = k). $$

iv) (Discrete = continuous) *n year pure endowment insurance* makes unit payment at time n only if $t > n$, otherwise no payment is made. Now

$$ v_t = \begin{cases} v^t, & t \le n \\ v^n, & t > n, \end{cases} \quad b_t = \begin{cases} 0, & t \le n \\ 1, & t > n \end{cases} \quad \text{and} \quad z_t = b_t v_t = \begin{cases} 0, & t \le n \\ v^n, & t > n. \end{cases} $$

The present value random variable

$$ Z_{\substack{1 \\ x:\overline{n}|}} = \begin{cases} 0, & T \le n \\ v^n, & T > n. \end{cases} $$

Then the actuarial present value APV = EPV = NSP =

$$ A_{\substack{1 \\ x:\overline{n}|}} = E(Z_{\substack{1 \\ x:\overline{n}|}}) = {}_nE_x = v^n P(T > n) = v^n \int_n^{\infty} f_T(t)\, dt = v^n \int_n^{\infty} {}_tp_x\, \mu_{x+t}\, dt = v^n\, {}_np_x, \text{ and} $$

$$ {}^2A_{\substack{1 \\ x:\overline{n}|}} = E[(Z_{\substack{1 \\ x:\overline{n}|}})^2] = v^{2n} P(T_x > n) = v^{2n} \int_n^{\infty} f_T(t)\, dt = v^{2n} \int_n^{\infty} {}_tp_x\, \mu_{x+t}\, dt = v^{2n}\, {}_np_x. $$

The 1 above the $\overline{n}|$ means unit benefit is payable after (x) dies if death is after time n.

Also $V(Z_{\substack{1 \\ x:\overline{n}|}}) = v^{2n}\, {}_np_x\, {}_nq_x$.

Note the book does not use $\overline{Z}$ and $\overline{A}$ for this insurance because payment is made iff $T_x > n$ iff $K_x > n$ so the discrete insurance and continuous insurance are technically equivalent.

98) The relationship between whole life insurance and n year temporary and n year deferred insurance is

$$ Z_x = Z^1_{x:\overline{n}|} + {}_n|Z_x, $$

$$ A_x = A^1_{x:\overline{n}|} + {}_n|A_x, $$

$$ [Z_x]^2 = [Z^1_{x:\overline{n}|}]^2 + [{}_n|Z_x]^2, \quad \text{and} $$

$$ {}^2A_x = {}^2A^1_{x:\overline{n}|} + {}^2\, {}_n|A_x. $$

99) Suppose (x) buys insurance and dies at $t \in (k-1, k]$ years from purchase so $K_x = k$ where $k \in \{1, 2, 3, ...\}$. Given a small table of k and $P(K_x = k)$, be able to find the following quantities. (Discrete) *n year endowment life insurance* makes unit payment at time $t = k$ if $t < k < n$ and at time n if $t > n$. Then $b_t = 1, t \ge 0$ and

$$ v_t = \begin{cases} v^t, & t \le n \\ v^n, & t > n \end{cases} \quad \text{and} \quad z_t = b_t v_t = \begin{cases} v^t, & t \le n \\ v^n, & t > n. \end{cases} $$

The present value random variable

$$Z_{x:\overline{n}|} = \begin{cases} v^{K_x}, & K_x \le n \\ v^n, & K_x > n. \end{cases}$$

Note that the n year endowment present value random variable $Z_{x:\overline{n}|} = Z^1_{x:\overline{n}|} + Z_{x:\overset{1}{\overline{n}|}}$, the sum of the n year term and n year pure endowment present value RVs.

Then the actuarial present value APV = EPV = NSP =

$$A_{x:\overline{n}|} = E[Z_{x:\overline{n}|}] = A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}} = \sum_{k=1}^{n} v^k P(K_x = k) + v^n P(K_x > n)$$

$$= \sum_{k=1}^{n} v^k P(K_x = k) + v^n \sum_{k=n+1}^{\infty} P(K_x = k).$$

Similarly, $[Z_{x:\overline{n}|}]^2 = [Z^1_{x:\overline{n}|}]^2 + [Z_{x:\overset{1}{\overline{n}|}}]^2$ and ${}^2A_{x:\overline{n}|} = {}^2A^1_{x:\overline{n}|} + {}^2A_{x:\overset{1}{\overline{n}|}}$

$$= \sum_{k=1}^{n} v^{2k} P(K_x = k) + v^{2n} P(K_x > n) =$$

$$\sum_{k=1}^{n} v^{2k} P(K_x = k) + v^{2n} \sum_{k=n+1}^{\infty} P(K_x = k).$$

100) Suppose (x) buys insurance and dies at $t > 0$ years from purchase so $T = T_x = t$. Given v, i or δ and the distribution of $T = T_x$, be able to find the following quantities for the following 5 continuous life insurance models where a unit payment (eg of \$100000, \$500000 or \$1000000) is made. Recall $v = \frac{1}{1+i} = e^{-\delta}$ and $\delta = \log(1+i) = -\log(v)$. Often use $v^t = e^{-\delta t}$ and $v^{2t} = e^{-2\delta t}$.

The rule of moments for $b_t \in \{0, 1\}$ (unit payment insurance) is if $E[\,\overline{Z}\,] = \overline{A} = g(\delta)$, then $E[(\overline{Z})^j] = {}^j\overline{A} = g(j\delta)$. This rule is usually used for $j = 2$.

i) (Continuous) *whole life insurance* makes unit payment at time $t = k$ with $v_t = v^t, t \ge 0$ and $b_t = 1, t \ge 0$. Then $z_t = b_t v_t = v^t, t \ge 0$. The present value random variable $\overline{Z}_x = z_T = v^T$. Then the actuarial present value APV = EPV = NSP =

$$\overline{A}_x = E(\overline{Z}_x) = E(v^T) = E(e^{-\delta T}) = \int_0^\infty v^t f_T(t)\, dt = \int_0^\infty e^{-\delta t} f_T(t)\, dt = \int_0^\infty v^t\, {}_tp_x\, \mu_{x+t}\, dt, \text{ and}$$

$${}^2\overline{A}_x = E[(\overline{Z}_x)^2] = E[(v^T)^2] = E(e^{-2\delta T}) = \int_0^\infty v^{2t} f_T(t)\, dt = \int_0^\infty e^{-2\delta t} f_T(t)\, dt = \int_0^\infty v^{2t}\, {}_tp_x\, \mu_{x+t}\, dt.$$

ii) (Continuous) *n year term insurance* makes unit payment at time $t > 0$ only if $t \le n$, otherwise no payment is made. Now $v_t = v^t, t \ge 0$,

$$b_t = \begin{cases} 1, & t \le n \\ 0, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} v^t, & t \le n \\ 0, & t > n. \end{cases}$$

The present value random variable

$$\overline{Z}^{1}_{x:\overline{n}|} = \begin{cases} v^{T_x}, & T \le n \\ 0, & T > n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$$\overline{A}^{1}_{x:\overline{n}|} = E(\overline{Z}^{1}_{x:\overline{n}|}) = \int_0^n e^{-\delta t} f_T(t)\, dt = \int_0^n v^t f_T(t)\, dt = \int_0^n v^t \ {}_tp_x \ \mu_{x+t}\, dt, \text{ and}$$

$${}^2\overline{A}^{1}_{x:\overline{n}|} = E[(\overline{Z}^{1}_{x:\overline{n}|})^2] = \int_0^n e^{-2\delta t} f_T(t)\, dt = \int_0^n v^{2t} f_T(t)\, dt = \int_0^n v^{2t} \ {}_tp_x \ \mu_{x+t}\, dt.$$

The 1 above the x means unit benefit is payable after (x) dies if death is not after time n.

iii) (Continuous) *n year deferred insurance* makes unit payment at time $t > 0$ only if $t > n$, otherwise no payment is made. Now $v_t = v^t, t \ge 0$,

$$b_t = \begin{cases} 0, & t \le n \\ 1, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} 0, & t \le n \\ v^t, & t > n. \end{cases}$$

The present value random variable

$${}_{n}|\overline{Z}_x = \begin{cases} 0, & T \le n \\ v^T, & T > n. \end{cases}$$

Then the actuarial present value APV = EPV = NSP =

$${}_{n}|\overline{A}_x = E({}_{n}|\overline{Z}_x) = \int_n^\infty e^{-\delta t} f_T(t)\, dt = \int_n^\infty v^t f_T(t)\, dt = \int_n^\infty v^t \ {}_tp_x \ \mu_{x+t}\, dt, \text{ and}$$

$${}^2\ {}_{n}|\overline{A}_x = E[({}_{n}|\overline{Z}_x)^2] = \int_n^\infty e^{-2\delta t} f_T(t)\, dt = \int_n^\infty v^{2t} f_T(t)\, dt = \int_n^\infty v^{2t} \ {}_tp_x \ \mu_{x+t}\, dt.$$

iv) See 97 iv) for the n year pure endowment life insurance which is both continuous and discrete.

v) (Continuous) *n year endowment life insurance* makes unit payment at time $t > 0$ if $t < n$ and at time n if $t > n$. Then $b_t = 1, t \ge 0$ and

$$v_t = \begin{cases} v^t, & t \le n \\ v^n, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} v^t, & t \le n \\ v^n, & t > n. \end{cases}$$

The present value random variable

$$\overline{Z}_{x:\overline{n}|} = \begin{cases} v^T, & T \le n \\ v^n, & T > n. \end{cases}$$

Note that the n year endowment present value random variable $\overline{Z}_{x:\overline{n}|} = \overline{Z}^{1}_{x:\overline{n}|} + Z_{\underset{x:\overline{n}|}{1}}$, the sum of the n year term and n year pure endowment present value RVs.

Then the actuarial present value APV = EPV = NSP =

$$\overline{A}_{x:\overline{n}|} = E[\overline{Z}_{x:\overline{n}|}] = \overline{A}^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1} = \int_0^n v^t f_T(t)\, dt + v^n P(T > n) = \int_0^n v^t\ {}_tp_x\ \mu_{x+t}\, dt + v^n\ {}_np_x.$$

Similarly, $[\overline{Z}_{x:\overline{n}|}]^2 = [\overline{Z}^1_{x:\overline{n}|}]^2 + [Z_{x:\overline{n}|}^{\ \ 1}]^2$ and ${}^2\overline{A}_{x:\overline{n}|} = {}^2\overline{A}^1_{x:\overline{n}|} + {}^2A_{x:\overline{n}|}^{\ \ 1}$

$$= \int_0^n v^{2t} f_T(t)\ dt + v^{2n} P(T_x > n) = \int_0^n v^{2t}\ {}_tp_x\ \mu_{x+t}\ dt\ +\ v^{2n}\ {}_np_x.$$

101) **Know**: Often $T_0 = X \sim EXP(\mu)$ so $T = T_x \sim EXP(\mu)$. This distribution occurs if the force of mortality μ, μ_x or μ_{x+t} is constant. Also $S_T(t) = {}_tp_x = e^{-\mu t}$. Hence $f_T(t) = {}_tp_x \mu_{x+t} = \mu e^{-\mu t}$.

102) **Know**: Often $T_0 = X \sim U(0, \omega)$ so $T_x \sim U(0, \omega - x)$. The uniform distribution has cdf that is linear and increases from 0 to 1 on its support. Its survival function is linear and decreases from 1 to 0 on its support. Hence l_x is linear and decreases from l_0 to 0 on its support. So $S(t) = 1 - t/\omega$ for $0 \le t \le \omega$, and ${}_tp_x = 1 - t/(\omega - x) = \dfrac{\omega - x - t}{\omega - x}$ for $0 \le t \le \omega - x$. Also $\mu_{x+t} = \dfrac{1}{\omega - x - t}$ and $f_T(t) = {}_tp_x \mu_{x+t} = \dfrac{1}{\omega - x}$ for $0 \le t < \omega - x$.

103) On SOA and CAS exams, often the notation A and Z is used even though the correct notation is $\overline{A}$ and $\overline{Z}$.

104) Whole life insurance with the exponential(μ) distribution often has $\overline{Z} = b_T\ v^T$ where $b_t = e^{\theta t}$. Now $\int_0^\infty \mu e^{-\mu t} dt = 1$ so $\int_0^\infty e^{-\mu t} dt = 1/\mu$ if $\mu > 0$. Hence $E[\overline{Z}\ ] = \int_0^\infty b_t e^{-\delta t} \mu e^{-\mu t} dt = \int_0^\infty e^{\theta t} e^{-\delta t} \mu e^{-\mu t} dt = \mu \int_0^\infty e^{-t[\mu + \delta - \theta]} dt = \dfrac{\mu}{\mu + \delta - \theta}$ provided $\mu + \delta - \theta > 0$. Also $E[(\overline{Z})^j] = \int_0^\infty [b_t e^{-\delta t}]^j \mu e^{-\mu t} dt = \int_0^\infty e^{\theta j t} e^{-\delta j t} \mu e^{-\mu t} dt = \mu \int_0^\infty e^{-t[\mu + \delta j - \theta j]} dt = \dfrac{\mu}{\mu + \delta j - \theta j}$ provided $\mu + \delta j - \theta j > 0$. Note that $\theta = 0$ corresponds to unit payment.

105) In 104), often $\int_0^\infty$ is replaced by $\int_a^b$.

106) For whole life insurance let ξ_α be the α percentile of $\overline{Z}$ so $P(\overline{Z} \le \xi_\alpha) = \alpha$ where $0 < \alpha < 1$. Assume unit payment so $\overline{Z} = v^T = e^{-\delta T}$. To find the α percentile ξ_α of $\overline{Z}$, solve $\alpha = P(\overline{Z} \le \xi_\alpha) = P(e^{-\delta T} \le \xi_\alpha) = P[-\delta T \le \log(\xi_\alpha)] = P\left(T \ge \dfrac{\log(\xi_\alpha)}{-\delta}\right) = S_T\left(\dfrac{-\log(\xi_\alpha)}{\delta}\right)$. So solve $\alpha = S_T\left(\dfrac{-\log(\xi_\alpha)}{\delta}\right)$ for ξ_α. Often $T \sim EXP(\mu)$ so $S_T(t) = e^{-\mu t}$.

107) On SOA and CAS exams, often a table of the curtate duration at failure random variable $K(x)$ and $\tilde{k}$ is given instead of a table of K_x and k. Since $K_x = K(x) + 1$, $K(x) = \tilde{k}$ means $K_x = k = \tilde{k} + 1$. So add 1 to each $\tilde{k}$ value and use the formula for K_x or note that $Z = z_{K_x} = z_{K(x)+1} = b_{K(x)+1} v^{K(x)+1}$ (provided $v_t = v^t$) and $A = E(Z)$.

108) Often unit benefits are not used. Let $Z = B_x = b_{K_x} v^{K_x}$ so the APV $= E[Z] = E[B_x] = \sum_{k=1}^\infty b_k v^k P(K_x = k)$ and $E[Z^j] = \sum_{k=1}^\infty (b_k v^k)^j P(K_x = k)$.

Use point 107) if a table of $K(x)$ and $\tilde{k}$ is given instead of a table of K_x and k.

109) Suppose (x) buys insurance and dies at time $t \in (k-1, k]$ after purchase.
a) (Discrete annually or unit) increasing whole life insurance pays k units at time k and has $v_t = v^t$ and $b_t = t$ for $t \geq 0$. So $z_t = b_t v_t = t v^t$ and the present value RV $Z = B_x = z_{K_x} = K_x v^{K_x}$. Hence the APV $= E(Z) = E[B_x] = (IA)_x = \sum_{k=1}^{\infty} k v^k P(K_x = k)$.

b) (Discrete annually or unit) increasing n year term insurance pays k units at time k if $k \leq n$ and 0 payment if $k > n$. So $v_t = v^t$ for $t \geq 0$ and

$$b_t = \begin{cases} t, & t \leq n \\ 0, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} tv^t, & t \leq n \\ 0, & t > n \end{cases}$$

and the present value RV $Z = z_{K_x}$. Hence the APV $= E(Z) = (IA)^1_{x:\overline{n}|} = \sum_{k=1}^{n} k v^k P(K_x = k)$.

110) Suppose (x) buys insurance and dies at time $t \in (k-1, k]$ after purchase. (Discrete annually or unit) decreasing n year term insurance pays $n-k+1$ units at time k if $t \leq n$ and 0 payment if $t > n$. So $v_t = v^t$ for $t \geq 0$ and

$$b_t = \begin{cases} n-t+1, & t \leq n \\ 0, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} (n-t+1)v^t, & t \leq n \\ 0, & t > n \end{cases}$$

and the present value RV $Z = z_{K_x}$. Hence the APV $=$
$E(Z) = (DA)^1_{x:\overline{n}|} = \sum_{k=1}^{n} (n-k+1) v^k P(K_x = k)$.

111) Often unit benefits are not used for continuous insurance. Let $\overline{B}_x = \overline{Z} = z_{T_x} = b_{T_x} v_{T_x}$. Then ${}^j\overline{A} = E[(\overline{Z})^j] = \int_0^{\infty} (b_t v_t)^j f_T(t) dt$. Note that APV $= \overline{A} = E[\overline{Z}] = E[\overline{B}_x] = \int_0^{\infty} b_t v_t f_T(t) dt$. The bars on A and Z are often omitted. Usually $v_t = v^t = e^{-\delta t}$.

112) Suppose (x) buys insurance and dies at time t after purchase.
a) (Continuous) increasing whole life insurance pays t units at time t and has $v_t = v^t$ and $b_t = t$ for $t \geq 0$. So $z_t = b_t v_t = t v^t$ and the present value RV $\overline{Z} = \overline{B}_x = z_{T_x} = T_x v^{T_x}$. Hence the APV $= E(\overline{Z}) = E[\overline{B}_x] = (\overline{I}\,\overline{A})_x = \int_0^{\infty} t v^t f_T(t) dt = \int_0^{\infty} t v^t \; {}_tp_x \; \mu_{x+t} \; dt$.

b) (Continuous) increasing n year term insurance pays t units at time t if $t \leq n$ and 0 payment if $t > n$. So $v_t = v^t$ for $t \geq 0$ and

$$b_t = \begin{cases} t, & t \leq n \\ 0, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} tv^t, & t \leq n \\ 0, & t > n \end{cases}$$

and the present value RV $\overline{Z} = z_{T_x}$. Hence the APV $= E(\overline{Z}) = (\overline{I}\,\overline{A})^1_{x:\overline{n}|} = \int_0^{n} t v^t f_T(t) dt = \int_0^{n} t v^t \; {}_tp_x \; \mu_{x+t} \; dt$.

c) (Continuous) decreasing n year term insurance pays $n-t$ units at time t if $t \leq n$ and 0 payment if $t > n$. So $v_t = v^t$ for $t \geq 0$ and

$$b_t = \begin{cases} n-t, & t \leq n \\ 0, & t > n \end{cases} \quad \text{and} \quad \mathrm{z_t = b_t v_t} = \begin{cases} (n-t)v^t, & t \leq n \\ 0, & t > n \end{cases}$$

and the present value RV $\overline{Z} = z_{T_x}$. Hence the APV $= E(\overline{Z}) = (\overline{D}\ \overline{A})^1_{x:\overline{n}|} =$
$\int_0^n (n-t)v^t f_T(t)dt = \int_0^n (n-t)v^t\ {}_tp_x\ \mu_{x+t}\ dt.$

Chapter 6

Life annuities = contingent annuity models make a series of payments contingent on the survival of (x). Life insurance models tend to make a single payment contingent on the death of (x).

113) Several quantities from interest theory (ch. 1) are useful. The quantities i, δ and v are the same as in ch. 5. So $v = 1/(1+i) = e^{-\delta}$ and $\delta = \log(1+i) = -\log(v)$.
New quantities are $d = \dfrac{i}{1+i} = iv = 1 - v > 0$,

$a_{\overline{n}|} = v + v^2 + v^3 + \cdots + v^n = \sum_{j=1}^{n} v^j = \dfrac{1-v^n}{i}$,

$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + v^3 + \cdots + v^{n-1} = \sum_{j=0}^{n-1} v^j = \dfrac{1-v^n}{d}$, and

$\overline{a}_{\overline{n}|} = \int_0^n v^t dt = \dfrac{1-v^n}{\delta}$.

114) A (discrete annual) immediate whole life annuity pays (x) 1 unit at times $t = 1, 2, ...,$ as long as (x) survives. For integer t, $P(K_x > t) = P(T_x > t) = {}_tp_x$. Let $Y_x = a_{\overline{K_x - 1}|}$ be the present value random variable and let $a_x = E(Y_x)$ be the APV = EPV = NSP of the annuity. Then $a_x = E(Y_x) = \sum_{t=1}^{\infty} v^t\ {}_tp_x$.

115) $Y_x = a_{\overline{K_x-1}|} = \dfrac{1 - v^{K_x - 1}}{i} = \dfrac{1}{i}[1 - (1+i)Z_x]$.

116) Given i and a table of y and l_y where $l_y = 0$ for $y > \omega$ and $y = x, x+1, ..., \omega$, find $E(Y_x)$ and $V(Y_x)$ using the following steps.

i) Convert the table into a table of t and ${}_tp_x$ using ${}_tp_x = \dfrac{l_{x+t}}{l_x}$.

ii) $E(Y_x) = a_x = \sum_{t=1}^{\omega} v^t\ {}_tp_x$.

iii) $A_x = v - d a_x$.

iv) ${}^2a_x = \sum_{t=1}^{\omega} (v^2)^t\ {}_tp_x$.

v) Let $i' = (1+i)^2 - 1$, let $v' = \dfrac{1}{(1+i)^2} = \dfrac{1}{1+i'}$, and let $d' = i'v' = \dfrac{i'}{1+i'}$
(these are d, v and i evaluated using a force of interest $\delta' = 2\delta$). Then

$${}^2A_x = v' - d'\ ({}^2a_x).$$

vi) $V(Y_x) = \dfrac{{}^2A_x - (A_x)^2}{d^2}$.

117) A (discrete annual) whole life annuity-due pays (x) 1 unit at times $t = 0, 1, 2, ...,$ as long as (x) survives. Let $\ddot{Y}_x = \ddot{a}_{\overline{K_x}|} = Y_x + 1$ be the present value random variable and let $\ddot{a}_x = E(\ddot{Y}_x)$ be the APV = EPV = NSP of the annuity. Then $\ddot{a}_x = E(\ddot{Y}_x) = a_x + 1 = \sum_{t=0}^{\infty} v^t\ {}_tp_x$. Note that $v^0 = 1$ and ${}_0p_x = 1$.

118) To find $E(\ddot{Y}_x)$ and $V(\ddot{Y}_x)$, complete steps i)-vi) of point 116). Then $E(\ddot{Y}_x) = a_x + 1$ and $V(\ddot{Y}_x) = V(Y_x) = \dfrac{{}^2A_x - (A_x)^2}{d^2}$ or use $A_x = 1 - d(\ddot{a}_x)$ and ${}^2A_x = 1 - d'({}^2\ddot{a}_x)$.

119) A continuous annuity can not exist but is a good approximation for a $m$thly annuity if $m \geq 12$.

120) A continuous whole life annuity makes a continuous payment at an annual rate of 1 unit per year as long as (x) survives. The present value RV $\overline{Y}_x = \overline{a}_{\overline{T_x}|} = \dfrac{1 - v^{T_x}}{\delta}$. The APV is $\overline{a}_x = E(\overline{Y}_x) = \int_0^\infty v^t \; {}_tp_x dt = \int_0^\infty v^t \; S_T(t) dt$ where $S_T(t) = S(x+t)/S(x)$.

121) A continuous whole life annuity makes a continuous payment at annual rate of 1 unit a year as long as (x) survives. If (x) dies at time t, then the present value is $\overline{a}_{\overline{t}|}$. The present value RV is $\overline{Y}_x = \overline{a}_{\overline{T_x}|} = \dfrac{1 - v^{T_x}}{\delta} = \dfrac{1 - \overline{Z}_x}{\delta}$. The APV is $\overline{a}_x = E[\overline{Y}_x] = \int_0^\infty v^t \; {}_tp_x \; dt = \int_0^\infty e^{-\delta t} S_T(t) dt$ where $S_T(t) = \dfrac{S(x+t)}{S(x)}$. $V(\overline{Y}_x) = \dfrac{V(\overline{Z}_x)}{\delta^2} = \dfrac{{}^2\overline{A}_x - (\overline{A}_x)^2}{\delta^2}$.

122) The (discrete) immediate n year temporary annuity pays (x) 1 unit at times $t = 1, ..., n$ if $K_x > n$ and at times $t = 1, ..., k-1$ if $2 \leq K_x = k \leq n$. No payment is made if $K_x = 1$. The present value RV

$$Y_{x:\overline{n}|} = \sum_{t=1}^{n} Z_{\overset{1}{x:\overline{t}|}} = \begin{cases} a_{\overline{K_x - 1}|}, & K_x \leq n \\ a_{\overline{n}|}, & K_x > n. \end{cases}$$

The APV $a_{x:\overline{n}|} = E(Y_{x:\overline{n}|}) = \sum_{t=1}^{n} v^t \; {}_tp_x$. The variance formula is complicated.

123) The (discrete) n year temporary annuity-due pays (x) 1 unit at times $t = 0, 1, ..., n-1$ if $K_x > n$ and at times $t = 0, 1, ..., k-1$ if $K_x = k \leq n$. The present value RV $\ddot{Y}_{x:\overline{n}|} = \sum_{t=0}^{n-1} Z_{\overset{1}{x:\overline{t}|}} = Y_{x:\overline{n}|} + 1 - Z_{x:\overline{n}|}$. The APV $\ddot{a}_{x:\overline{n}|} = E(\ddot{Y}_{x:\overline{n}|}) = \sum_{t=0}^{n-1} v^t \; {}_tp_x = \dfrac{1 - A_{x:\overline{n}|}}{d}$ $= a_{x:\overline{n}|} + 1 - \; {}_nE_x$ and $V(\ddot{Y}_{x:\overline{n}|}) = \dfrac{V(Z_{x:\overline{n}|})}{d^2} = \dfrac{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}{d^2}$ where $A_{x:\overline{n}|} = 1 - d(\ddot{a}_{x:\overline{n}|})$, ${}^2A_{x:\overline{n}|} = 1 - d'({}^2\ddot{a}_{x:\overline{n}|})$, and ${}^2\ddot{a}_{x:\overline{n}|} = \sum_{t=0}^{n-1} v^{2t} \; {}_tp_x$.

124) A (continuous) temporary n year annuity makes a continuous payment at annual rate of 1 unit a year for n years if $T_x > n$ and for T_x years if $T_x < n$. The present value RV is

$$\overline{Y}_{x:\overline{n}|} = \frac{1 - \overline{Z}_{x:\overline{n}|}}{\delta} = \begin{cases} \overline{a}_{\overline{T_x}|}, & T_x \leq n \\ \overline{a}_{\overline{n}|}, & T_x > n. \end{cases}$$

The APV is $\overline{a}_{x:\overline{n}|} = E(\overline{Y}_{x:\overline{n}|}) = \int_0^n v^t \; {}_tp_x \; dt = \int_0^n e^{-\delta t} S_T(t) dt$ where $S_T(t) = \dfrac{S(x+t)}{S(x)}$. $V(\overline{Y}_{x:\overline{n}|}) = \dfrac{{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2}{\delta^2}$ where $\overline{A}_{x:\overline{n}|} = 1 - \delta(\overline{a}_{x:\overline{n}|})$, ${}^2\overline{A}_{x:\overline{n}|} = 1 - \delta'({}^2\overline{a}_{x:\overline{n}|})$, $\delta' = 2\delta$ and ${}^2\overline{a}_{x:\overline{n}|} = \int_0^n v^{2t} \; {}_tp_x \; dt = \int_0^n e^{-2\delta t} S_T(t) dt$.

125) A (discrete) immediate n year deferred whole life annuity makes no payment if $K_x \leq n+1$. If $K_x = k \geq n+2$, then unit payment is made at times $t = n+1, n+2, ..., k-1$. The present value RV $_{n}|Y_x = Y_x - Y_{x:\overline{n}|} = \sum_{t=n+1}^{\infty} Z_{x:\overline{t}|}^{\;\;1}$. The APV $_{n}|a_x = E(_{n}|Y_x) = a_x - a_{x:\overline{n}|} = \sum_{t=n+1}^{\infty} v^t \; _{t}p_x$.

126) A (discrete) n year deferred whole life annuity-due makes no payment if $K_x \leq n$. If $K_x = k \geq n+1$, then unit payment is made at times $t = n, n+1, n+2, ..., k-1$. The present value RV $_{n}|\ddot{Y}_x = \ddot{Y}_x - \ddot{Y}_{x:\overline{n}|} = Z_{x:\overline{n}|}^{\;\;1} + \; _{n}|Y_x = \sum_{t=n}^{\infty} Z_{x:\overline{t}|}^{\;\;1}$. The APV $_{n}|\ddot{a}_x = E(_{n}|\ddot{Y}_x) = v^n \; _{n}p_x + \; _{n}|a_x = \sum_{t=n}^{\infty} v^t \; _{t}p_x$.

127) A (continuous) n year deferred annuity makes no payment if $T_x \leq n$. If $T_x = t > n$ then continuous payment at annual unit rate is made from time n to time t. The present value RV is

$$_{n}|\overline{Y}_x = \overline{Y}_x - \overline{Y}_{x:\overline{n}|} = \begin{cases} 0, & T_x \leq n \\ v^n \; \overline{a}_{\overline{T_x - n}|}, & T_x > n. \end{cases}$$

The APV $_{n}|\overline{a}_x = E(_{n}|\overline{Y}_x) = \overline{a}_x - \overline{a}_{x:\overline{n}|} = \int_n^\infty v^t \; _{t}p_x \; dt = \int_n^\infty e^{-\delta t} S_T(t) dt$.

$E[(_{n}|\overline{Y}_x)^2] = \frac{2}{\delta} \, v^{2n} \; _{n}p_x \; [\overline{a}_{x+n} - \, ^2\overline{a}_{x+n}]$ where $\overline{a}_{x+n} = \int_0^\infty e^{-\delta t} \; _{t}p_{x+n} \; dt = \int_0^\infty e^{-\delta t} S_{T_{x+n}}(t) dt$ and $^2\overline{a}_{x+n} = \int_0^\infty e^{-2\delta t} \; _{t}p_{x+n} \; dt = \int_0^\infty e^{-2\delta t} S_{T_{x+n}}(t) dt$.

128) Contingent annuities paid $m$thly are paid m times a year with payment $1/m$ where $m \geq 1$. So annual payment is 1 unit per year. Let $J_x = j$ denote death in the jth $m$thly time interval. Let i be the effective $m$thly interest rate (eg 1% per month if $m = 12$) and $i^{(m)}$ be the annual interest rate. Let $Z_x^{(m)} = v^{J_x}$ and $A_x^{(m)} = E(Z_x^{(m)}) = \sum_{j=1}^{\infty} v^j P(J_x = j)$. Let $V(Z_x^{(m)}) = \; ^2A_x^{(m)} - (A_x^{(m)})^2$ where $^2A_x^{(m)} = \sum_{j=1}^{\infty} v^{2j} P(J_x = j)$.

129) A discrete immediate $m$thly whole life annuity pays $1/m$ units at the end of each $m$thly time interval while (x) survives. So if $J_x = j$, the 1st payment is at time $t = 1/m$ and the last payment is made at the end of the $(j-1)$th $m$thly interval. The present value RV is $Y_x^{(m)} = \frac{1}{m} a_{\overline{J_x - 1}|} = \frac{1}{m} \frac{1 - v^{J_x - 1}}{i}$. The APV is

$$a_x^{(m)} = E[Y_x^{(m)}] = \frac{1}{m} \sum_{t=1}^{\infty} v^{\frac{t}{m}} \; _{\frac{t}{m}}p_x \quad \text{and} \quad \mathrm{V}(\mathrm{Y}_\mathrm{x}^{(\mathrm{m})}) = \frac{^2\mathrm{A}_\mathrm{x}^{(\mathrm{m})} - (\mathrm{A}_\mathrm{x}^{(\mathrm{m})})^2}{\mathrm{m}^2 \mathrm{d}^2}.$$

130) A discrete $m$thly whole life annuity-due pays $1/m$ units at the beginning of each $m$thly time interval while (x) survives. So if $J_x = j$, the 1st payment is at time $t = 0$ and the last payment is made at the beginning of the jth $m$thly interval. The present

value RV is $\ddot{Y}_x^{(m)} = \frac{1}{m}\ddot{a}_{\overline{J_x}|} = Y_x^{(m)} + \frac{1}{m}$. The APV is

$$\ddot{a}_x^{(m)} = a_x^{(m)} + \frac{1}{m} = E[\ddot{Y}_x^{(m)}] = \frac{1}{m}\sum_{t=0}^{\infty} v^{\frac{t}{m}} \; {}_{\frac{t}{m}}p_x \text{ and } \mathrm{V}(\ddot{\mathrm{Y}}_\mathrm{x}^{(\mathrm{m})}) = \mathrm{V}(\mathrm{Y}_\mathrm{x}^{(\mathrm{m})}).$$

131) **KNOW:** Let $T \sim EXP(\mu)$. Then $E(T) = \int_0^\infty t\mu e^{-\mu t} dt = \int_0^\infty e^{-\mu t} dt = 1/\mu$. So $\int_0^\infty tDe^{-t(D)} dt = \int_0^\infty e^{-t(D)} dt = 1/D$ for $D > 0$. Use $\stackrel{E}{=}$ when exponential RV is used.

132) **KNOW:** Let $T \sim EXP(\mu)$. $S(t) = e^{-\mu t}$ for $t > 0$. Often use Z instead of $\overline{Z}$.

i) If $b_t = ce^{\theta t}$ and $Z = b_T v_T$, then $E[Z^j] = E[(b_T v_T)^j] = c^j E[(e^{\theta T} v_T)^j]$. So multiply $c = 1$ formulas by c^j. Usually want $j = 1, 2$.

a) Special whole life insurance: $b_t = e^{\theta t}$, $v_t = e^{-\delta t}$, and $Z = b_T v_T = e^{\theta T} e^{-\delta T}$. $E(Z^j) = \dfrac{\mu}{\mu + \delta j - \theta j}$ if $\mu + \delta j - \theta j$. See 104).

b) Whole life insurance: special case of a) with $\theta = 0$. See 100i). $\overline{Z}_x = e^{-\delta T}$. $\overline{A}_x = E(\overline{Z}_x) = E(e^{-\delta T}) \stackrel{E}{=} \dfrac{\mu}{\mu + \delta}$, and ${}^2\overline{A}_x = E[(\overline{Z}_x)^2] = E(e^{-2\delta T}) \stackrel{E}{=} \dfrac{\mu}{\mu + 2\delta}$.
$V(\overline{Z}_x) = {}^2\overline{A}_x - (\overline{A}_x)^2$.

c) Whole life annuity. See 121). $\overline{Y}_x = \dfrac{1 - \overline{Z}_x}{\delta}$.
$E[\overline{Y}_x] = \overline{a}_x = \int_0^\infty e^{-\delta t} S_T(t) dt \stackrel{E}{=} \dfrac{1}{\mu + \delta}$. $\quad V(\overline{Y}_x) = \dfrac{V(\overline{Z}_x)}{\delta^2} = \dfrac{{}^2\overline{A}_x - (\overline{A}_x)^2}{\delta^2}$.

133) If $D > 0$, $\int_0^n De^{-tD} dt = 1 - e^{-nD}$, $\int_n^\infty De^{-tD} dt = e^{-nD}$,
$\int_0^n e^{-tD} dt = \dfrac{1}{D}[1 - e^{-nD}]$, and $\int_n^\infty e^{-tD} dt = \dfrac{1}{D} \, e^{-nD}$.