Math 402 Exam 2 is Wed. March. 29. **You are allowed 12 sheets of notes and a calculator.** The exam emphasizes HW4-7, and Q4-7. 1) - 38) on Exam 1 review will be useful.

**ch. 9.** In multiple decrement models, $(x)$ is subject to multiple contingencies where each type of failure is called a decrement. There are $m$ distinct causes of failure. On exams, usually $m = 2$ and double decrement models are used. Symbols with the superscript $(\tau)$ are similar to those used for life tables.

74) **Know:** Given $x$, $q_x^{(1)}$, $q_x^{(2)}$ and the initial group size $= l_x^{(\tau)}$ for the smallest $x$ in the table, be able to fill in the double decrement table ($m = 2$) with headers shown below. A useful fact is $l_{x+1}^{(\tau)} = l_x^{(\tau)} - d_x^{(1)} - d_x^{(2)}$ except for rounding. If $m > 2$, then there are more columns, $q_x^{(\tau)} = q_x^{(1)} + \cdots + q_x^{(m)}$, and $l_{x+1}^{(\tau)} = l_x^{(\tau)} - d_x^{(1)} - \cdots - d_x^{(m)}$ except for rounding.

| x | $q_x^{(1)}$ | $q_x^{(2)}$ | $q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)}$ | $p_x^{(\tau)} = 1 - q_x^{(\tau)}$ | $l_x^{(\tau)} = p_{x-1}^{(\tau)}\, l_{x-1}^{(\tau)}$ | $d_x^{(1)} = l_x^{(\tau)}\, q_x^{(1)}$ | $d_x^{(2)} = l_x^{(\tau)}\, q_x^{(2)}$ |
|---|---|---|---|---|---|---|---|

75) **Know:** Use the above table to find the following quantities.

i) $q_x^{(j)} = P[(x)$ fails in the next year due to the $j$th cause$]$.

ii) $q_x^{(\tau)} = \sum_{j=1}^m q_x^{(j)} = P[(x)$ fails in the next year$]$.

iii) $p_x^{(\tau)} = 1 - q_x^{(\tau)} = P[(x)$ does not fail in the next year$]$.

iv) $d_x^{(j)} = l_x^{(\tau)} q_x^{(j)} =$ (expected) number of people in group at age $x$ who will fail before age $x + 1$ due to cause $j$.

v) $d_x^{(\tau)} = l_x^{(\tau)} q_x^{(\tau)} = \sum_{j=1}^m d_x^{(j)} =$ (expected) number of people in group at age $x$ who will fail before age $x + 1$.

vi) $l_x^{(\tau)} = \sum_{j=1}^m l_x^{(j)} =$ (expected) total number in the group at age $x$.

vii) $l_x^{(j)} =$ (expected) number in the group at age $x$ who eventually fail due to cause $j$.

For viii)-xii), note that $n = 1$ is usually omitted, so ${}_1q_x^{(\tau)} = q_x^{(\tau)}$, et cetera.

viii) $${}_nq_x^{(j)} = \frac{{}_nd_x^{(j)}}{l_x^{(\tau)}} = \frac{\sum_{t=0}^{n-1} d_{x+t}^{(j)}}{l_x^{(\tau)}} = \text{P( of failure due to cause } j \text{ in } (x, x+n])$$

ix) $${}_nq_x^{(\tau)} = \sum_{j=1}^m {}_nq_x^{(j)} = \text{P( of failure in } (x, x+n])$$

x) $${}_np_x^{(\tau)} = \frac{l_{x+n}^{(\tau)}}{l_x^{(\tau)}} = 1 - {}_nq_x^{(\tau)} = \text{P( of survival in } (x, x+n])$$

xi) ${}_nd_x^{(j)} = l_x^{(\tau)}\, {}_nq_x^{(j)} = \sum_{t=0}^{n-1} d_{x+t}^{(j)} =$ (expected) number who fail due to cause $j$ in $(x, x+n]$

xii) ${}_nd_x^{(\tau)} = l_x^{(\tau)}\, {}_nq_x^{(\tau)} = \sum_{j=1}^m {}_nd_x^{(j)} =$ (expected) number who fail in $(x, x+n]$

xiii) P$[(x)$ will die between $x + n$ and $x + n + m$ due to cause $j] = {}_{n|m}q_x^{(j)} = \dfrac{d_{x+n}^{(j)} + \cdots + d_{x+n+m-1}^{(j)}}{l_x^{(\tau)}}$ so ${}_{k|}q_x^{(j)} = \dfrac{d_{x+k}^{(j)}}{l_x^{(\tau)}} = p_{K_x,J_x}(k, j)$. (Note $m = 1$ is suppressed.)

xiv) $${}_{k|}q_x^{(\tau)} = {}_kp_x^{(\tau)} - {}_{k+1}p_x^{(\tau)} = \frac{l_{x+k}^{(\tau)}}{l_x^{(\tau)}} - \frac{l_{x+k+1}^{(\tau)}}{l_x^{(\tau)}} = \frac{d_{x+k}^{(\tau)}}{l_x^{(\tau)}} = \frac{\sum_{j=1}^m d_{x+k}^{(j)}}{l_x^{(\tau)}} = P(K_x = k)$$

76) Some more life table formulas:

a) $l_{x+1}^{(\tau)} = l_x^{(\tau)} - d_x^{(\tau)}$

b) $l_{x+k}^{(\tau)} = l_x^{(\tau)}\, {}_kp_x^{(\tau)}$

c) $d_{x+k}^{(\tau)} = l_x^{(\tau)} \ {}_k p_x^{(\tau)} \ q_{x+k}^{(\tau)} = l_x^{(\tau)} \ {}_{k|} q_x^{(\tau)}$

d) $d_{x+k}^{(j)} = l_x^{(\tau)} \ {}_k p_x^{(\tau)} \ q_{x+k}^{(j)} = l_x^{(\tau)} \ {}_{k|} q_x^{(j)}$

77) A useful fact is ${}_n p_x^{(\tau)} = p_x^{(\tau)} p_{x+1}^{(\tau)} \cdots p_{x+n-1}^{(\tau)}$.

78) $K_x = k$ means $(x)$ fails in the $(k+1)$th interval (year) and $J_x = j$ means failure was due to the $j$th cause. The joint probability function of $K_x$ and $J_x$ is $p_{K_x,J_x}(k,j) = P(K_x = k, J_x = j) = {}_{k|} q_x^{(j)} = \dfrac{d_{x+k}^{(j)}}{l_x^{(\tau)}}$ given by point 75xiii).

79) The marginal probability function of $K_x$ is $p_{K_x}(k) = P(K_x = k) = {}_{k|} q_x^{(\tau)} = \dfrac{\sum_{j=1}^m d_{x+k}^{(j)}}{l_x^{(\tau)}}$ given by point 75xiv). Note that the numerator corresponds to the sum of the $d_{(j)}$ values in the row of the multiple decrement table corresponding to age $x+k$.

80) The marginal probability function of $J_x$ is $p_{J_x}(j) = P(J_x = j) = \dfrac{\sum_{k=0}^\infty d_{x+k}^{(j)}}{l_x^{(\tau)}}$. Note that the numerator corresponds to the sum of the $d_{(j)}$ values in the column of the multiple decrement table corresponding to cause $j$.

81) Let $T_x^{(j)}$ be the time until failure RV for $(x)$ due to cause $j$ in the absence of all other $m-1$ decrements. Then $T_x^{(j)}$ is like $T_x$ in ch. 2-5. A prime will be used in the actuarial notation for $p$ and $q$ but not for $\mu$ in the absence of all other $m-1$ decrements. Assume $t > 0$. The rules from point 5) still hold.

i) cdf: $F_{T_x^{(j)}}(t) = P(T_x^{(j)} < t) = {}_t q_x^{'(j)} = 1 - {}_t p_x^{'(j)} = \int_0^t f_{T_x^{(j)}}(s) ds = \int_0^t {}_s p_x^{'(j)} \mu_{x+s}^{(j)} ds$

ii) survival function: $S_{T_x^{(j)}}(t) = {}_t p_x^{'(j)} = 1 - {}_t q_x^{'(j)} = P(T_x^{(j)} > t) = \exp[-\int_0^t \mu_{x+s}^{(j)} \ ds]$.

iii) force of failure: $\mu_{T_x^{(j)}}(t) = \mu_{x+t}^{(j)} = \dfrac{-\frac{d}{dt} S_{T_x^{(j)}}(t)}{S_{T_x^{(j)}}(t)} = \dfrac{-\frac{d}{dt} \ {}_t p_x^{'(j)}}{{}_t p_x^{'(j)}} = \dfrac{f_{T_x^{(j)}}(t)}{S_{T_x^{(j)}}(t)}$

iv) pdf $f_{T_x^{(j)}}(t) = S_{T_x^{(j)}}(t) \ \mu_{x+t}^{(j)} = {}_t p_x^{'(j)} \ \mu_{x+t}^{(j)} = \dfrac{d}{dt} F_{T_x^{(j)}}(t) = -\dfrac{d}{dt} S_{T_x^{(j)}}(t)$

82) Consider the time until failure RV for $(x)$ when all $m$ decrements are present. A $\tau$ will be used in the actuarial notation for $p$, $q$ and $\mu$. The rules from point 5) still hold, but there are special equations for ${}_t p_x^{(\tau)}$ and $\mu_{x+t}^{(\tau)}$ using terms from 81).

i) cdf: ${}_t q_x^{(\tau)} = 1 - {}_t p_x^{(\tau)} = \int_0^t {}_s p_x^{(\tau)} \mu_{x+s}^{(\tau)} ds$

ii) survival function: ${}_t p_x^{(\tau)} = 1 - {}_t q_x^{(\tau)} = \exp[-\int_0^t \mu_{x+s}^{(\tau)} \ ds] = \prod_{j=1}^m {}_t p_x^{'(j)}$.

iii) force of failure: $\mu_{x+t}^{(\tau)} = \dfrac{-\frac{d}{dt} \ {}_t p_x^{(\tau)}}{{}_t p_x^{(\tau)}} = \sum_{j=1}^m \mu_{x+t}^{(j)}$

83) If cause $j$ was the only cause of failure (decrement), then the absolute rate of decrement due to cause $j$ over $(x, x+n]$ is ${}_n q_x^{'(j)} =$ P(failing due to cause $j$ in $(x, x+n]$) if no other of the $m-1$ causes of failure (decrements) were acting. Also, ${}_1 q_x^{'(j)} = q_x^{'(j)}$.

84) Recall that ${}_n q_x^{(j)} =$ P(failing due to cause $j$ in $(x, x+n]$) when there are $m$ causes of decrement, and ${}_1 q_x^{(j)} = q_x^{(j)}$.

85) The following three quantities are found when there are $m$ causes of decrement. Note that these quantities do not have a prime ' in the superscript.

i) ${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(j)} ds$ = P(of failing to cause $j$ in $(x, x+t]$) when there are $m$ causes of decrement.

ii) ${}_\infty q_x^{(j)} = \lim_{t\to\infty} {}_tq_x^{(j)}$ = P(of failing to cause $j$ eventually) when there are $m$ causes of decrement.

iii) $\mu_{x+t}^{(j)} = \dfrac{\frac{d}{dt}\, {}_tq_x^{(j)}}{{}_tp_x^{(\tau)}}$ [Also see 81 iii).]

Note that the superscript $(\tau)$ is used for all $m$ causes of failure and corresponds to the RV "$W = T_x$", the superscript $'(j)$ corresponds to RV $T_x^{(j)}$ which is failure from the $j$th cause when none of the other $m-1$ causes of failure (decrements) are present, and the superscript $(j)$ corresponds to failure due to cause $j$ when there are $m$ causes of failure. An exception is $\mu_{x+t}^{(j)} = \mu_{x+t}^{'(j)}$, so actuarial notation omits the prime. Another exception is $T_x^{(j)}$ does not have a prime. These 3 superscripts are used for continuous multiple decrement models.

86) a) ${}_tq_x^{(\tau)} = \sum_{j=1}^{m} {}_tq_x^{(j)}$ = P[ $(x)$ fails in the next $t$ years].

b) ${}_{t|}q_x^{(j)} = {}_tp_x^{(\tau)} q_{x+t}^{(j)}$ = P[ $(x)$ fails between $x+t$ and $x+t+1$ due to cause $j$].

c) ${}_{t|m}q_x^{(j)} = \sum_{k=t}^{t+m-1} {}_kp_x^{(\tau)} q_{x+k}^{(j)} =$
P[ $(x)$ fails between $x+t$ and $x+t+m$ due to cause $j$].

87) The $l_x^{(\tau)}$ are sometimes called **active lives.** Hence if decrements are (d) for death, (w) for withdrawal, (i) for disability and (r) for retirement, as for the **illustrative service table**, then a life is **not active** at age $x+t$ if the person died, withdrew, had a disability, or retired in the interval $(x, x+t]$. Be able to use the illustrative service table.

88) The probability that someone succumbed (became inactive) due to cause $j$ at time $t$, given that someone succumbed (to some decrement) at time $t$ is $\dfrac{\mu_{x+t}^{(j)}}{\mu_{x+t}^{(\tau)}}$.

89) The prime $'$ denotes a single decrement quantity. To go from multiple to single decrement quantities i) find ${}_tp_x^{(\tau)} = 1 - \sum_{j=1}^{m} {}_tq_x^{(j)}$. ii) Find $\mu_{x+t}^{(j)} = \dfrac{\frac{d}{dt}\, {}_tq_x^{(j)}}{{}_tp_x^{(\tau)}}$, and iii) find ${}_tp_x^{'(j)} = \exp[-\int_0^t \mu_{x+s}^{(j)}\, ds]$.

90) To go from single to multiple decrement quantities i) find $\mu_{x+t}^{(j)} = \dfrac{\frac{d}{dt}\, {}_tq_x^{'(j)}}{{}_tp_x^{'(j)}} = \dfrac{\frac{-d}{dt}\, {}_tp_x^{'(j)}}{{}_tp_x^{'(j)}}$. ii) Find ${}_tp_x^{(\tau)} = \prod_{j=1}^{m} {}_tp_x^{'(j)} = \exp[-\int_0^t \mu_{x+s}^{(\tau)}\, ds]$. iii) Find ${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(j)} ds$.

∮ 10.7

91) For the common shock model (x) and (y) are dependent and are exposed to a common hazard, called a common shock. Let $\mu_{x+t}^*$ be the forces of failure specific to (x) but not to (y) at time $t$. Let $\mu_{y+t}^*$ be the forces of failure specific to (y) but not to (x) at time $t$. Let the common hazard = common shock for both (x) and (y) be a constant

$\mu_t^C \equiv \lambda$ for $t \geq 0$.

92) An equivalent way to develop the common shock model is to let $T_x^*$ and $T_y^*$ denote the future lifetime random variables for (x) and (y) without regard to the common shock hazard function. Let $W \sim EXP(\lambda)$ denote the future lifetime of either (x) or (y) with regard to the common shock hazard factors only. Assume $T_x^* \perp\!\!\!\perp T_Y^* \perp\!\!\!\perp W$. Let $\mu_{x+t}^*$ be the force of failure for $T_x^*$, and $\mu_{y+t}^*$ that of $T_y^*$. Then the survival function $S_{T_x^*}(t) = {}_tp_x^* = \exp(-\int_0^t \mu_{x+r}^* dr)$ and the survival function $S_{T_y^*}(t) = {}_tp_y^* = \exp[-\int_0^t \mu_{y+r}^* dr]$. Then ${}_tp_x = \exp[-\int_0^t (\mu_{x+r}^* + \lambda)dr]$, ${}_tp_y = \exp[-\int_0^t (\mu_{y+r}^* + \lambda)dr]$, and
${}_tp_{xy} = \exp[-\int_0^t (\mu_{x+r}^* + \mu_{y+r}^* + \lambda)dr]$.

93) **Know** For the common shock model,
i) $\mu_{x+t} = \mu_{x+t}^* + \lambda, \quad {}_tp_x = {}_tp_x^* \ e^{-\lambda t}$;
ii) $\mu_{y+t} = \mu_{y+t}^* + \lambda, \quad {}_tp_y = {}_tp_y^* \ e^{-\lambda t}$;
iii) the total force of failure for the joint life status (xy) is
$\mu_{xy}(t) = \mu_{T_{xy}}(t) = \mu_{x+t:y+t} = \mu_{x+t}^* + \mu_{y+t}^* + \lambda = \mu_{x+t} + \mu_{y+t} - \lambda, \ {}_tp_{xy} = {}_tp_x^* \ {}_tp_y^* \ e^{-\lambda t} = {}_tp_x \ {}_tp_y \ e^{\lambda t}$;
iv) $P(T_x = T_y) = \int_0^\infty {}_tp_{xy} \ \lambda dt$.

94) **Know** Chapter 10 formulas still hold for the common shock model, but $T_x$ is not independent of $T_y$. So for the last survivor status $(\overline{xy})$,
i) $T_{xy} + T_{\overline{xy}} = T_x + T_y$. Also $g(T_{xy}) + g(T_{\overline{xy}}) = g(T_x) + g(T_y)$, and $T_{\overline{xy}} = T_x + T_y - T_{xy}$.
ii) survival function: $S_{T_{\overline{xy}}}(t) = {}_tp_{\overline{xy}} = P(T_{\overline{xy}} > t) = {}_tp_x + {}_tp_y - {}_tp_{xy}$.
iii) cdf: $F_{T_{\overline{xy}}}(t) = {}_tq_{\overline{xy}} = P(T_{\overline{xy}} \leq t) = 1 - S_{T_{\overline{xy}}}(t)$
iv) $\overset{o}{e}_{\overline{xy}} = E(T_{\overline{xy}}) = \int_0^\infty t f_{T_{\overline{xy}}}(t)dt = \int_0^\infty {}_tp_{\overline{xy}} \ dt = \overset{o}{e}_x + \overset{o}{e}_y - \overset{o}{e}_{xy}$.

v) $\overline{A}_{\overline{xy}} = \overline{A}_x + \overline{A}_y - \overline{A}_{xy}$.
vi) $\overline{a}_{\overline{xy}} = \overline{a}_x + \overline{a}_y - \overline{a}_{xy}$.

95) **Know** Suppose $T_x^* \sim EXP(\mu_x^*)$ and $T_y^* \sim EXP(\mu_y^*)$ in the common shock model (so $W \sim EXP(\lambda)$). Then i) $T_x \sim EXP(\mu_x = \mu_x^* + \lambda)$,
ii) $T_y \sim EXP(\mu_y = \mu_y^* + \lambda)$,
iii) $T_{xy} \sim EXP(\mu_x^* + \mu_y^* + \lambda = \mu_x + \mu_y - \lambda)$,
iv) $\overline{A}_x = \dfrac{\mu_x^* + \lambda}{\mu_x^* + \lambda + \delta}, \quad \overline{A}_y = \dfrac{\mu_y^* + \lambda}{\mu_y^* + \lambda + \delta}, \quad \overline{A}_{xy} = \dfrac{\mu_x^* + \mu_y^* + \lambda}{\mu_x^* + \mu_y^* + \lambda + \delta}$.
v) $\overline{a}_x = \dfrac{1}{\mu_x^* + \lambda + \delta}, \quad \overline{a}_y = \dfrac{1}{\mu_y^* + \lambda + \delta}, \quad \overline{a}_{xy} = \dfrac{1}{\mu_x^* + \mu_y^* + \lambda + \delta}$.
vi) $P(T_x = T_y) = \dfrac{\lambda}{\mu_x^* + \mu_y^* + \lambda}$.

**Warning:** Could be given $T_x \sim EXP(\mu_x)$, $T_y \sim EXP(\mu_y)$, and $\lambda$. Then $\mu_x^* = \mu_x - \lambda$ and $\mu_y^* = \mu_y - \lambda$. Suppose you are told that the force of mortalities for (x) and (y) are $u$ and $v$. If you are told *common shock is incorporated* into the forces of mortality $u$ and $v$, then $u = \mu_{x+t}$ and $v = \mu_{y+t}$. If you are told the *noncommon forces of mortality* for (x) and (y) are $u$ and $v$, then $u = \mu_{x+t}^*$ and $v = \mu_{y+t}^*$. End ∮ 10.7 material

**Some ch. 10 material from Exam 1 review:**

41) A **joint life status** for $(xy)$ fails as soon as $x$ or $y$ dies. Let $T_{xy} = \min(T_x, T_y) =$ time until 1st death. Convert $q$'s to $p$'s, then convert back to $q$'s if needed.

43) **Know:** Consider a joint life status $(xy)$ and $T_{xy}$.

i) survival function: $S_{xy}(t) = {}_tp_{xy} = P(T_{xy} > t)$. If $T_x \perp\!\!\!\perp T_y$, then ${}_tp_{xy} = ({}_tp_x)({}_tp_y)$.

ii) cdf: $F_{xy}(t) = {}_tq_{xy} = P(T_{xy} \leq t)$. If $T_x \perp\!\!\!\perp T_y$, then ${}_tq_{xy} = {}_tq_x + {}_tq_y - ({}_tq_x)({}_tq_y)$.

iv) force of mortality: $\mu_{xy}(t) = \dfrac{f_{xy}(t)}{S_{xy}(t)}$. If $T_x \perp\!\!\!\perp T_y$, then $\mu_{xy}(t) = \mu_{x+t} + \mu_{y+t} \equiv \mu_{x+t:y+t}$.

vi) $\overset{o}{e}_{xy} = E(T_{xy}) = \int_0^\infty t f_{xy}(t) dt = \int_0^\infty {}_tp_{xy} dt$.

45) If $T_x \sim EXP(\mu_x) \perp\!\!\!\perp T_y \sim EXP(\mu_y)$, then $T_{xy} = \min(T_x, T_y) \sim EXP(\mu_x + \mu_y)$.

46) A two life **last survivor status** for $(\overline{xy})$ fails after both $x$ and $y$ die. Let $T_{\overline{xy}} = \max(T_x, T_y) =$ time until 2nd death. Then $T_{xy} + T_{\overline{xy}} = T_x + T_y$. Convert $p$'s to $q$'s, then convert back to $p$'s if needed.

48) **Know:** Consider a last survivor status $(\overline{xy})$ and $T_{\overline{xy}}$.

i) survival function: $S_{T_{\overline{xy}}}(t) = S_{\overline{xy}}(t) = {}_tp_{\overline{xy}} = P(T_{\overline{xy}} > t) = {}_tp_x + {}_tp_y - {}_tp_{xy}$. If $T_x \perp\!\!\!\perp T_y$, then ${}_tp_{\overline{xy}} = 1 - ({}_tq_x)({}_tq_y) = {}_tp_x + {}_tp_y - ({}_tp_x)({}_tp_y)$.

ii) cdf: $F_{\overline{xy}}(t) = {}_tq_{\overline{xy}} = P(T_{\overline{xy}} \leq t) = 1 - S_{\overline{xy}}(t)$. If $T_x \perp\!\!\!\perp T_y$, then ${}_tq_{\overline{xy}} = ({}_tq_x)({}_tq_y) = F_x(t)F_y(t) = F_{T_x}(t)F_{T_y}(t)$.

vi) $\overset{o}{e}_{\overline{xy}} = E(T_{\overline{xy}}) = \int_0^\infty t f_{\overline{xy}}(t) dt = \int_0^\infty {}_tp_{\overline{xy}}\, dt = \overset{o}{e}_x + \overset{o}{e}_y - \overset{o}{e}_{xy}$.

50) $T_{xy}$ is one of $T_x$ or $T_y$, and $T_{\overline{xy}}$ is the other. Hence $T_{xy} + T_{\overline{xy}} = T_x + T_y$, and $T_{\overline{xy}} = T_x + T_y - T_{xy}$. Similarly, $P(T_{xy} > t) + P(T_{\overline{xy}} > t) = P(T_x > t) + P(T_y > t)$, and $P(T_{\overline{xy}} > t) = P(T_x > t) + P(T_y > t) - P(T_{xy} > t)$. See point 48) i) and vi).

72) **Know:** Let $T_{x_1}, ..., T_{x_m}$ be independent EXP($\mu_i$) RVs. Let $u = (x_1 \cdots x_m)$ or $u = x_1 \cdots x_m$. Then $T = T_u = T_{x_1 \cdots x_m} = \min(T_{x_1}, ..., T_{x_m}) \sim EXP(\sum_{i=1}^m \mu_i)$. Then $\mu_T(t) = \sum_{i=1}^m \mu_i$, $S_T(t) = \exp(-t \sum_{i=1}^m \mu_i)$, $\overset{o}{e}_u = E(T) = 1/(\sum_{i=1}^m \mu_i)$ and $V(T) = 1/(\sum_{i=1}^m \mu_i)^2$. a) For whole life insurance, $\overline{A}_u = E[\overline{Z}_u] = \dfrac{\sum_{i=1}^m \mu_i}{\delta + \sum_{i=1}^m \mu_i}$, and ${}^2\overline{A}_u = E[(\overline{Z}_u)^2] = \dfrac{\sum_{i=1}^m \mu_i}{2\delta + \sum_{i=1}^m \mu_i}$.

b) For a whole life annuity, $\overline{a}_u = E[\overline{Y}_u] = \dfrac{1}{\delta + \sum_{i=1}^m \mu_i}$, and $V[\overline{Y}_u] = \dfrac{{}^2\overline{A}_u - (\overline{A}_u)^2}{\delta^2}$.

**chapter 18**

96) The **indicator function** $I_A(x) \equiv I(x \in A) = 1$ if $x \in A$ and 0, otherwise. Sometimes an indicator function such as $I_{(0,\infty)}(y)$ will be denoted by $I(y > 0)$.

97) If none of the survival times are censored or truncated, then the **empirical survival function** = (number of individual with survival times $> t$)/(number of individuals) $= n_t/n = \hat{S}_E(t) = \frac{1}{n}\sum_{i=1}^{n} I(T_i > t) = \hat{p}_t$ = sample proportion of lifetimes $> t$.

Let $t_{(1)} \le t_{(2)} \le \cdots \le t_{(n)}$ be the observed ordered survival times (= lifetimes = death times). Let $t_0 = 0$ and let $0 < t_1 < t_2 < \cdots < t_m$ be the distinct survival times. Let $d_i$ = number of deaths at time $t_i$. If $m = n$ and $d_i = 1$ for $i = 1, ..., n$ then there are **no ties**. If $m < n$ and some $d_i \ge 2$, then there are **ties**.

$\hat{S}_E(t)$ is a step function with $\hat{S}_E(0) = 1$ and $\hat{S}_E(t) = \hat{S}_E(t_{i-1})$ for $t_{i-1} \le t < t_i$. Note that $\sum_{i=1}^{m} d_i = n$.

98) **know**: A linear or Wald 95% confidence interval (CI) for $\theta$ is $\hat{\theta} \pm 1.96\ SE(\hat{\theta})$.

99) **know**: Let $\hat{S}_E(t) = n_t/n$, as in 97). Then

$$SE(\hat{S}_E(t)) = \hat{S}_E(t)\sqrt{\frac{1}{n_t} - \frac{1}{n}}.$$

Thus a 95% CI for $S(t)$ is $\hat{S}_E(t) \pm 1.96\ SE(\hat{S}_E(t))$.

100) **know**: Suppose $n$ people aged $x$ buy a $t$ year life insurance policy. Then the estimated number of claims that will be filed is

$$n\left[1 - \frac{\hat{S}(t+x)}{\hat{S}(x)}\right].$$

101) **know**:

| interval | length $L_i$ | deaths $d_i$ | $\hat{S}_x(\sum_{j=1}^{i}(L_i))$ |
|---|---|---|---|
| $[x, x_1)$ | $L_1$ | $d_1$ | $\hat{S}_x(L_1) = \frac{n-d_1}{n} = \hat{S}_x(0) - \frac{d_1}{n}$ |
| $[x_1, x_2)$ | $L_2$ | $d_2$ | $\hat{S}_x(L_1+L_2) = \frac{n-(d_1+d_2)}{n} = \hat{S}_x(L_1) - \frac{d_2}{n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $[x_{i-1}, x_i)$ | $L_i$ | $d_i$ | $\hat{S}_x(\sum_{j=1}^{i} L_j) = \frac{n-(d_1+d_2+\cdots+d_i)}{n} = \hat{S}_x(\sum_{j=1}^{i-1} L_j) - \frac{d_i}{n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $[x_{k-1}, x_k)$ | $L_k$ | $d_k$ | $\hat{S}_x(\sum_{j=1}^{k} L_j) = \frac{n-(d_1+d_2+\cdots+d_k)}{n} = \hat{S}_x(\sum_{j=1}^{k-1} L_j) - \frac{d_k}{n}$ |

Suppose we have $n$ lives observed from exact age $x$ to $x_k$ as tabled below. Then $\hat{S}_x(0) = 1$, $\hat{S}_x(L_1) = \frac{n-d_1}{n} = \hat{S}_x(0) - \frac{d_1}{n}$, $\hat{S}_x(\sum_{j=1}^{i} L_j) = \frac{n-(d_1+d_2+\cdots+d_i)}{n} = \hat{S}_x(\sum_{j=1}^{i-1} L_j) - \frac{d_i}{n}$. Linear interpolation is used to find the *ogive empirical survival function*

$$\hat{S}_X(t) = \frac{(t_U - t)\hat{S}_x(t_L) + (t - t_L)\hat{S}_x(t_U)}{t_U - t_L}$$

for $t_L \leq t < t_U$.

Know how to compute $\hat{S}_x(t)$ with a table like the one above. The second and third columns need to be given.

102) Let $Y_i^* = Y_i + = T_i = \min(Y_i, Z_i)$ where $Y_i$ and $Z_i$ are independent. Let $\delta_i = I(Y_i \leq Z_i)$ so $\delta_i = 1$ if $T_i$ is uncensored and $\delta_i = 0$ if $T_i$ is censored. Let $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$ be the observed ordered survival times. Let $\gamma_j = 1$ if $t_{(j)}$ is uncensored and 0, otherwise. Let $t_0 = 0$ and let $0 < t_1 < t_2 < \cdots < t_m$ be the distinct survival times corresponding to the $t_{(j)}$ with $\gamma_j = 1$. Let $d_i$ = number of deaths at time $t_i$. If $m = n$ and $d_i = 1$ for $i = 1, ..., n$ then there are **no ties**. If $m < n$ and some $d_i \geq 2$, then there are **ties**.

103)

| $t_i$ | $r_i$ | $d_i$ | $\hat{S}_K(t)$ |
|---|---|---|---|
| $t_0 = 0$ | | | $\hat{S}_K(0) = 1$ |
| $t_1$ | $r_1$ | $d_1$ | $\hat{S}_K(t_1) = \hat{S}_K(t_0)[1 - \frac{d_1}{r_1}]$ |
| $t_2$ | $r_2$ | $d_2$ | $\hat{S}_K(t_2) = \hat{S}_K(t_1)[1 - \frac{d_2}{r_2}]$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $t_j$ | $r_j$ | $d_j$ | $\hat{S}_K(t_j) = \hat{S}_K(t_{j-1})[1 - \frac{d_j}{r_j}]$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $t_{m-1}$ | $r_{m-1}$ | $d_{m-1}$ | $\hat{S}_K(t_{m-1}) = \hat{S}_K(t_{m-2})[1 - \frac{d_{m-1}}{r_{m-1}}]$ |
| $t_m$ | $r_m$ | $d_m$ | $\hat{S}_K(t_m) = 0 = \hat{S}_K(t_{m-1})[1 - \frac{d_m}{r_m}]$ |

**Know**: Let $r_i = \sum_{j=1}^{n} I(t_{(j)} \geq t_i) = \#$ at risk at $t_i = \#$ alive and not yet censored just before $t_i$. Let $d_i = \#$ of events (deaths) at $t_i$. The **Kaplan Meier estimator** of $S_Y(t_i) = P(Y > t_i)$ is $\hat{S}_K(0) = 1$ and $\hat{S}_K(t_i) = \prod_{k=1}^{i}(1 - \frac{d_k}{r_k}) = \hat{S}_K(t_{i-1})(1 - \frac{d_i}{r_i})$. $\hat{S}_K(t)$ is a step function with $\hat{S}_K(t) = \hat{S}_K(t_{i-1})$ for $t_{i-1} \leq t < t_i$ and $i = 1, ..., m$. If $t_{(n)}$ is uncensored then $t_m = t_{(n)}$ and $\hat{S}_K(t) = 0$ for $t > t_m$. If $t_{(n)}$ is censored, then $\hat{S}_K(t) = \hat{S}_K(t_m)$ for $t_m \leq t \leq t_{(n)}$, but $\hat{S}_K(t)$ is undefined for $t > t_{(n)}$.

Know how to compute $\hat{S}_k(t_i)$ given a) the $t_{(j)}$ and $\gamma_j$, or b) given the $t_i$, $n_i$ and $d_i$, or c) given a small data set.

104) **Know:** The Kaplan Meier estimator, given $t_j$, $d_j$, $c_j$, and $r_0$, can be computed for a big data set. Let $c_j$ = number of exits − number of new entrants in $[t_j, t_{j+1})$. Let the risk set at $t_j$ be $r_j$ = number of observed lives at risk just before time $t_j$ (so at time $t_j^-$). Let $r_0$, $t_j$, $d_j$ and $c_j$ be given. Then $r_1 = r_0 - c_0$ and $r_{j+1} = r_j - d_j - c_j$. Then we still have $\hat{S}_K(t_i) = \prod_{k=1}^{i}(1 - \frac{d_k}{r_k}) = \hat{S}_K(t_{i-1})(1 - \frac{d_i}{r_i})$. Given the first 3 columns and $r_0$, be able to fill in the last two columns of the table below.

| $t_j$ | $d_j$ | $c_j$ | $r_j$ | $\hat{S}_K(t)$ |
|---|---|---|---|---|
| $t_0 = 0$ | | $c_0$ | $r_0$ | $\hat{S}_K(0) = 1$ |
| $t_1$ | $d_1$ | $c_1$ | $r_1$ | $\hat{S}_K(t_1) = \hat{S}_K(t_0)[1 - \frac{d_1}{r_1}]$ |
| $t_2$ | $d_2$ | $c_2$ | $r_2$ | $\hat{S}_K(t_2) = \hat{S}_K(t_1)[1 - \frac{d_2}{r_2}]$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $t_j$ | $d_j$ | $c_j$ | $r_j$ | $\hat{S}_K(t_j) = \hat{S}_K(t_{j-1})[1 - \frac{d_j}{r_j}]$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $t_{m-1}$ | $d_{m-1}$ | $c_{m-1}$ | $r_{m-1}$ | $\hat{S}_K(t_{m-1}) = \hat{S}_K(t_{m-2})[1 - \frac{d_{m-1}}{r_{m-1}}]$ |
| $t_m$ | $d_m$ | $c_m$ | $r_m$ | $\hat{S}_K(t_m) = 0 = \hat{S}_K(t_{m-1})[1 - \frac{d_m}{r_m}]$ |

105) For both 103) and 104), Greenwood's formula is

$$\hat{V}(\hat{S}_K(t)) = [\hat{S}_K(t)]^2 \sum_{j:t_j \le t} \frac{d_j}{r_j(r_j - d_j)}$$

where $t_0$ is not used. Then $SE(\hat{S}_K(t)) = \sqrt{\hat{V}(\hat{S}_K(t))}$. For $t = t_i$ use the sum $\sum_{j=1}^{i}$.

106) The Nelson Aalon estimator of $H(t)$ is the step function

$$\hat{H}_N(t_i) = \sum_{j=1}^{i} \frac{d_j}{r_j} = \hat{H}_N(t_{i-1}) + \frac{d_i}{r_i}$$

with $\hat{H}_N(0) = 0$ and

$$\hat{V}(\hat{H}_N(t_i)) = \sum_{j=1}^{i} \frac{d_j(r_j - d_j)}{r_j^3}.$$

For general $t$, can replace $\sum_{j=1}^{i}$ by $\sum_{j:t_j \le t}$.

107) $\hat{S}_N(t) = e^{-\hat{H}_N(t)}$ is a step function.
$\hat{H}_K(t_i) = -\log(\hat{S}_K(t_i))$ is a step function with $\hat{H}_K(0) = 0$

**Chapter 6–Premiums**

In chapters 4 and 5, wanted to know the lump sum paid at time $t = 0$ the insured should pay for insurance or an annuity. The lump sum was equal to the APV $A$ or $a$. In this chapter the insured pays premiums $P$ at times 0, 1, ..., $w$. Hence the insured is paying the insurance company with an annuity-due (with payment $P$ made at the beginning of each year). Now want to know what should $P$ be. The **equivalence principle** says E(present value of premiums] = E[present value of benefits]. If $\ddot{a}$ is the unit APV that the insured pays the insurance company, then E(present value of premiums] = the annuity-due APV paid by the insured = $(P)(\ddot{a})$. E[present value of benefits] = APV of the insurance or deferred annuity paid by the insurance company to the insured's beneficiary. So the equivalence principle sets $(P)(\ddot{a})$ = insurance company APV. The premium payments $(P)(\ddot{a})$ are the funding used to pay for the insurance or deferred annuity. Funding should not extend beyond the event that triggers payment nor beyond $n$ years for $n$ year term insurance or $n$ year deferred annuity.

108) The loss random variable for discrete insurance is $L = Z - P\ddot{Y}$ and under the equivalence principle, $E(L) = 0$ so $P = \frac{A}{\ddot{a}}$. The subscripts on $P$, $L$ and $Z$ are the same, but $\ddot{a}_x$ is used if premium payment could be indefinite while $\ddot{a}_{x:\overline{m}|}$ is used if payment is for at most $m$ years where $m = n$ or $m = t < n$.

109) Now suppose the funding payment is made continously at rate (or with continuous premium) $\overline{P}$. Discrete insurance with continuous premium $\overline{P}$.

i) whole life: $\overline{P}_x = \frac{A_x}{\overline{a}_x}$.

ii) $n$ year term: $\overline{P}^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}}$.

iii) $n$ year pure endowment: $\overline{P}_{\underset{x:\overline{n}|}{\ \ 1}} = \frac{A_{\underset{x:\overline{n}|}{\ \ 1}}}{\overline{a}_{x:\overline{n}|}}$.

iv) $n$ year endowment: $\overline{P}_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}}$.

v) $n$ year deferred insurance: $\overline{P}(\ {}_n|A_x) = \frac{{}_n|A_x}{\overline{a}_{x:\overline{n}|}}$.

110) Continuous whole life insurance with continuous premium:

$$\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x} = \frac{1 - \delta\ \overline{a}_x}{\overline{a}_x} = \frac{1}{\overline{a}_x} - \delta = \frac{\delta\ \overline{A}_x}{1 - \overline{A}_x}.$$

$$\overline{L}(\overline{A}_x) = \overline{Z}_x - [\overline{P}(\overline{A}_x)]\overline{Y}_x.$$

$$V(\overline{L}(\overline{A}_x)) = \left(1 + \frac{[\overline{P}(\overline{A}_x)]}{\delta}\right)^2 [\ {}^2\overline{A}_x - (\overline{A}_x)^2] = \left(\frac{1}{\delta\ \overline{a}_x}\right)^2 [\ {}^2\overline{A}_x - (\overline{A}_x)^2] = \frac{{}^2\overline{A}_x - (\overline{A}_x)^2}{(1 - \overline{A}_x)^2}.$$

**Know:** If $T_x \sim EXP(\mu)$, then $\overline{P}(\overline{A}_x) = \mu$, and $V(\overline{L}(\overline{A}_x)) = \frac{\mu}{\mu + 2\delta} = {}^2\overline{A}_x$.

111) Continuous $n$ year endowment life insurance with continuous premium:

$\overline{P}(\overline{A}_{x:\overline{n}|}) = \dfrac{\overline{A}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} = \dfrac{1}{\overline{a}_{x:\overline{n}|}} - \delta = \dfrac{\delta\, \overline{A}_{x:\overline{n}|}}{1-\overline{A}_{x:\overline{n}|}}.$

$V(\overline{L}(\overline{A}_{x:\overline{n}|})) = \left(1 + \dfrac{[\overline{P}(\overline{A}_{x:\overline{n}|})]}{\delta}\right)^2 \; [\,{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2] = \dfrac{{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2}{(1-\overline{A}_{x:\overline{n}|})^2}.$

**Know:** If $T_x \sim EXP(\mu)$, then $\overline{P}(\overline{A}_{x:\overline{n}|}) = \mu$, and $V(\overline{L}(\overline{A}_{x:\overline{n}|})) = \dfrac{\mu}{\mu+2\delta}$.

Note that the difference between 110) and 111) is that the whole life insurance drops the $\overline{n}|$.

112) If the benefit is $b$ instead of 1, and $bP$ is the premium for $b$ units of insurance, multiply the variances (in 110) and 111)) by $b^2$.

113) Some other continuous insurances with continuous premium:

i) $n$ year term insurance: $\overline{P}(\overline{A}^1_{x:\overline{n}|}) = \dfrac{\overline{A}^1_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}}$

**Know:** If $T_x \sim EXP(\mu)$, then $\overline{P}(\overline{A}^1_{x:\overline{n}|}) = \mu$.

ii) $n$ year deferred insurance: ${}_n\overline{P}(\,{}_{n|}\overline{A}_x) = \dfrac{{}_{n|}\overline{A}_x}{\overline{a}_{x:\overline{n}|}}$

Here premiums are paid only during the $n$ year deferral period.

iii) $n$-pay whole life insurance: ${}_n\overline{P}(\overline{A}_x) = \dfrac{\overline{A}_x}{\overline{a}_{x:\overline{n}|}}$

Here benefit is payable at death and premium is payable for $n$ years.

iv) $n$ year deferred annuity: ${}_n\overline{P}(\,{}_{n|}\overline{a}_x) = \dfrac{{}_{n|}\overline{a}_x}{\overline{a}_{x:\overline{n}|}}$

114) $V(L) = \dfrac{{}^2A - (A)^2}{(1-A)^2}$ for $L = L_x, L_{x:\overline{n}|}, \overline{L}_x$, and $\overline{L}_{x:\overline{n}|}$ under the equivalence principle.

115) Discrete insurance under the equivalence principle.

i) whole life: $L_x = Z_x - P_x\ddot{Y}_x$ with $P_x = \dfrac{A_x}{\ddot{a}_x} = \dfrac{1}{\ddot{a}_x} - d = \dfrac{dA_x}{1-A_x}$.

ii) $n$ year term: $L^1_{x:\overline{n}|} = Z^1_{x:\overline{n}|} - P^1_{x:\overline{n}|}\ddot{Y}_{x:\overline{n}|}$ with $P^1_{x:\overline{n}|} = \dfrac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$.

iii) $n$ year pure endowment: $L_{\substack{\ \ 1\\ x:\overline{n}|}} = Z_{\substack{\ \ 1\\ x:\overline{n}|}} - P_{\substack{\ \ 1\\ x:\overline{n}|}}\ddot{Y}_{x:\overline{n}|}$ with $P_{\substack{\ \ 1\\ x:\overline{n}|}} = \dfrac{A_{\substack{\ \ 1\\ x:\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}$.

iv) $n$ year endowment: $L_{x:\overline{n}|} = Z_{x:\overline{n}|} - P_{x:\overline{n}|}\ddot{Y}_{x:\overline{n}|}$ with $P_{x:\overline{n}|} = \dfrac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \dfrac{1}{\ddot{a}_{x:\overline{n}|}} - d = \dfrac{dA_{x:\overline{n}|}}{1-A_{x:\overline{n}|}}$.

116) For the insurance models in 115), $L = Z - P\ddot{Y}$ and $\ddot{Y} = \dfrac{1-Z}{d}$. It can be shown that $V(L) = \left(1+\dfrac{P}{d}\right)^2 \; [\,{}^2A - (A)^2]$.

i) whole life: $V(L_x) = \left(1+\dfrac{P_x}{d}\right)^2 \; [\,{}^2A_x - (A_x)^2] = \left(\dfrac{1}{d\ddot{a}_x}\right)^2 \; [\,{}^2A_x - (A_x)^2] =$

$\dfrac{{}^2A_x - (A_x)^2}{(1-A_x)^2}$.

ii) $n$ year term: $V(L^1_{x:\overline{n}|}) = \left(1 + \dfrac{P^1_{x:\overline{n}|}}{d}\right)^2 \; [\,{}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2]$

iii) $n$ year pure endowment: $V(L_{x:\overline{n}|}^{\;\;1}) = \left(1 + \dfrac{P_{x:\overline{n}|}^{\;\;1}}{d}\right)^2 \; [\,{}^2A_{x:\overline{n}|}^{\;\;1} - (A_{x:\overline{n}|}^{\;\;1})^2]$

iv) $n$ year endowment: $V(L_{x:\overline{n}|}) = \left(1 + \dfrac{P_{x:\overline{n}|}}{d}\right)^2 \; [\,{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2] = \dfrac{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}{(1-A_{x:\overline{n}|})^2}$

117) The above variance formulas are for unit payment where $L \equiv L(1)$. For payment $X$, let $L(X)$ be the loss RV. Then $V[L(X)] = X^2 V[L(1)]$.

118) $P_{x:\overline{n}|} = P^1_{x:\overline{n}|} + P_{x:\overline{n}|}^{\;\;1}$

119) Limited payment of t-pay insurance funds the insurance for $t < n$ years.

i) limited payment (t-pay) whole life has ${}_tP_x = \dfrac{A_x}{\ddot{a}_{x:\overline{t}|}}$.

ii) limited payment (t-pay) $n$ year term has ${}_tP^1_{x:\overline{n}|} = \dfrac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}}$.

120) i) An $n$ year deferred insurance has $P(\,{}_n|A_x) = \dfrac{{}_n|A_x}{\ddot{a}_x}$.

ii) For $t < n$, a limited payment (t-pay) $n$ year deferred insurance has ${}_tP(\,{}_n|A_x) = \dfrac{{}_n|A_x}{\ddot{a}_{x:\overline{t}|}}$.

121) Suppose the continuous insurance pays the claim immediately but the premiums are paid annually.

i) whole life has $P(\overline{A}_x) = \dfrac{\overline{A}_x}{\ddot{a}_x}$.

ii) $n$ year term has $P(\overline{A}^1_{x:\overline{n}|}) = \dfrac{\overline{A}^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$.

iii) $n$ year endowment has $P(\overline{A}_{x:\overline{n}|}) = \dfrac{\overline{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$.

122) i) $n$ year deferred immediate annuity has $L = {}_n|Y_x - [P(\,{}_n|a_x)]\; \ddot{Y}_{x:\overline{n}|}$ with $P(\,{}_n|a_x) = \dfrac{{}_n|a_x}{\ddot{a}_{x:\overline{n}|}}$.

ii) $n$ year deferred annuity-due has $L = {}_n|\ddot{Y}_x - [P(\,{}_n|\ddot{a}_x)]\; \ddot{Y}_{x:\overline{n}|}$ with $P(\,{}_n|\ddot{a}_x) = \dfrac{{}_n|\ddot{a}_x}{\ddot{a}_{x:\overline{n}|}}$.

iii) $n$ year continuous deferred annuity $P(\;n|\overline{a}_x) = \dfrac{n|\overline{a}_x}{\ddot{a}_{x:\overline{n}|}}$.

123) **Know**: The illustrative life table is often useful for calculating premiums.

## Chapter 8
## Markov Chains

124) A (finite or finite state) *Markov chain* $\{X_n : n = 0, 1, 2, ...\}$ is a discrete stochastic process for which time only takes on integer values. $X_n$ will have $J$ possible values 1, ..., $J$ called states. If $X_n = i \in \{1, ..., J\}$, then the Markov chain is in state $i$ at time $n$. Suppose $x_k \in \{1, ..., J\}$ for $k \geq 0$. The *Markov property* is

$$P(X_{n+1} = j|X_n = i, X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, ..., X_1 = x_1, X_0 = x_0) = P(X_{n+1} = j|X_n = i)$$

for any $n \geq 1$. Hence the conditional probability of $X_{n+1}$ given the past only depends on the state the Markov chain is in at time $X_n$. Or, given $X_n = i$, then $X_{n+1}$ is independent of the rest of the past (time periods 0, 1, ..., $n-1$). If $0 \leq d < n$ then $P(X_{n+1} = j|X_n = i, X_{n-1} = x_{n-1}, ..., X_d = x_d) = P(X_{n+1} = j|X_n = i)$.

125) **Know:** The *transition probability* $p_{ij} = P(X_{n+1} = j|X_n = i)$. The *transition probability matrix*

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1J} \\ p_{21} & p_{22} & \cdots & p_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ p_{J1} & p_{J2} & \cdots & p_{JJ} \end{bmatrix}.$$

126) The sum of the probabilities in any row of $\boldsymbol{P}$ is $\sum_{j=1}^{J} p_{ij} = 1$ for row $i = 1, ..., J$.

127) For small $J$, a transition diagram list the $J$ states with $J$ arrows leaving each state and $J$ arrows entering each state. Then there are $J^2$ arrows corresponding to the $p_{ij}$ that form $\boldsymbol{P}$. An arrow labelled $p_{ij}$ goes from state $i$ to state $j$. An arrow labelled $p_{ii}$ goes from state $i$ to state $i$. A variant on the transition diagram leaves out $p_{ii}$, which can be found using 126), and leaves out any arrow corresponding to $p_{ij} = 0$ for $i \neq j$.

128) **Know:** $P(X_{m+n} = j|X_m = i) = p^n_{ij}$ where $p^n_{ij}$ is the $ij$th entry of $\boldsymbol{P}^n = \boldsymbol{P}\boldsymbol{P}\cdots\boldsymbol{P}$ where there are $n$ matrices $\boldsymbol{P}$ in the multiplication. This formula is for a homogeneous Markov chain where the transition probability matrix does not depend on the time period $j$, so $\boldsymbol{P} = \boldsymbol{P}^{(j)}$ for $j = 0, 1, 2, ....$

129) State $j$ is accessible from state $i$ if $p^n_{ij} > 0$ for some $n \geq 0$. Then, starting in state $i$, it is possible that the process will enter state $j$ in a finite number of steps.

130) Two states $i$ and $j$ that are accessible to each other *communicate*, written $i \leftrightarrow j$.

131) States that communicate with each other form an equivalence class. A Markov chain is *irreducible* if there is only one class, so all states communicate.

132) For state $i$, let $r_i$ denote the probability, starting in state $i$, that the process will ever reenter state $i$. State $i$ is recurrent if $r_i = 1$ and transient if $r_i < 1$. State $i$ is absorbing if $p_{ii} = 1$ so that the other entries in the $i$th row are 0. Once in an absorbing state, such as death, the Markov chain stays in the absorbing state. An absorbing state is recurrent. All of the states in an irreducible Markov chain are recurrent.

133) A recurrent state will be visited infinitely often. A transient state is not certain to be revisited and will only be visited a finite number of times. Hence a Markov chain

must have at least on recurrent state to run indefinitely for $n = 1, 2, ...$. Starting in a transient state $i$, the number of time periods $N$ the process will be in state $i$, including the initial time, is geometric with finite mean $E(N) = 1/(1 - r_i)$. State $i$ is recurrent if $E(N) = \infty$ and is transient if $E(N) < \infty$.

134) If state $i$ is recurrent and $i \leftrightarrow j$, then state $j$ is recurrent. If state $i$ is transient and $i \leftrightarrow j$, then state $j$ is transient.

135) **Know:** Let $\boldsymbol{\pi}_n = (\pi_{1n}, ..., \pi_{Jn})$ denote the vector of probabilities of being in states 1 to $J$ at time $n$. Let $\boldsymbol{\pi}_0 = (\pi_{10}, ..., \pi_{J0})$ where $\pi_{i0} = P(X_0 = i)$ is the probability that the process is in state $i$ at the start, time 0. Then $\boldsymbol{\pi}_n$ is the *state vector* at time $n$ and

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 \boldsymbol{P}^n = \boldsymbol{\pi}_1 \boldsymbol{P}^{n-1} = \boldsymbol{\pi}_2 \boldsymbol{P}^{n-2} = \cdots = \boldsymbol{\pi}_k \boldsymbol{P}^{n-k} = \cdots = \boldsymbol{\pi}_{n-1} \boldsymbol{P}$$

and $\boldsymbol{\pi}_{n+1} = \boldsymbol{\pi}_n \boldsymbol{P}$. This formula is for a homogeneous Markov chain.

136) $\boldsymbol{\pi}_0$ is the initial distribution of the Markov chain. Either $\boldsymbol{\pi}_0$ is given or the problem states that the Markov chain starts in state $j$. Then $\boldsymbol{\pi}_0 = (0, ..., 0, 1, 0, ..., 0)$ where the 1 is in position $j$.

137) **Know:** For a *nonhomogeneous Markov chain*, the matrix of transition probabilities $\boldsymbol{P}^{(k)}$ depends on the $k$th step of the process. Then $\boldsymbol{\pi}_n$ = state vector at time $n$ satisfies $\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 \boldsymbol{P}^{(1)} \boldsymbol{P}^{(2)} \cdots \boldsymbol{P}^{(n)}$.

Sometimes the following notation is used $\boldsymbol{P}^{(j)} = \boldsymbol{P}_j = \boldsymbol{Q}^{(j)} = \boldsymbol{Q}_j$.

138) **Know:** For hand calculations multiply the state vector times the matrix. Avoid multiplying matrices. So $\boldsymbol{\pi}_3 = (\boldsymbol{\pi}_0 \boldsymbol{P}^{(1)}) \boldsymbol{P}^{(2)} \boldsymbol{P}^{(3)} = (\boldsymbol{\pi}_1 \boldsymbol{P}^{(2)}) \boldsymbol{P}^{(3)} = \boldsymbol{\pi}_2 \boldsymbol{P}^{(3)}$ for a nonhomogeneous Markov chain, and $\boldsymbol{\pi}_3 = (\boldsymbol{\pi}_0 \boldsymbol{P}) \boldsymbol{P} \boldsymbol{P} = (\boldsymbol{\pi}_1 \boldsymbol{P}) \boldsymbol{P} = \boldsymbol{\pi}_2 \boldsymbol{P}$ for a homogeneous Markov chain.