Math 402 Final is Friday May 15, 8-10 AM. **You are allowed 20 sheets of notes and a calculator.** This handout gives material since exam 3 useful for the final, Q10-11, and HW10-11.

∮ 7.5

160) For a gross premium G, the equivalence principle says that $E({}_0L_e) = E({}_0L) =$ APV(benefits + expenses) − APV(gross premiums), where ${}_0L_e = {}_0L$ is the loss random variable at issue. One type of problem uses this equation to solve for G.

161) A variant of 160) is to find (the observed value of) ${}_0L_e = {}_0L =$ APV(benefits + expenses) − APV(gross premiums) for person who died in the kth year where k is small and G is given.

∮ 10.6

162) $${}_kAS = \frac{[\,{}_{k-1}AS + G(1-c_{k-1}) - e_{k-1}](1+i) - b_k q^{(d)}_{x+k-1} - {}_kCV q^{(w)}_{x+k-1}}{1 - q^{(d)}_{x+k-1} - q^{(w)}_{x+k-1}}$$

163) Given all of the variables in 162) except one, usually ${}_kAS$ or i, calculate the unknown variable where ${}_kAS$ is the asset share at the end of year k,
G is the gross premium (= contract premium),
c_k is the proportion of the premium payable as an expense at time k, starting at $k = 0$,
e_k is the per policy expense at time k,
b_k is the face amount,
$q^{(d)} = q^{(1)}$ is the death probability,
$q^{(w)} = q^{(2)}$ is the withdrawal probability,
${}_kCV$ is the cash value at time k.

164) Usually assume ${}_0AS = 0$.

165) The formula is 162) is for fully discrete insurance. So premiums are paid at the beginning of the year and benefits at the end of the year.

Percentiles and Probabilities for MLC Quantities ∮ 2.1, 7.6

166) **Know**: Let T be a random variable. If $P(T \le t_\alpha) = F_T(t_\alpha) = \alpha$, then t_α is the αth percentile of T. Similarly, $P(Z \le z_\alpha) = \alpha$ and $P(Y \le y_\alpha) = \alpha$. Often $t_\alpha = \xi_\alpha$ is used for the αth percentile.

167) **Know**: If g is an increasing function, then $g(t_\alpha)$ is the αth percentile of $g(T)$, while if g is a decreasing function, then $g(t_{1-\alpha})$ is the αth percentile of $g(T)$. Increasing and decreasing are "strictly increasing" and "strictly decreasing" in some texts.

168) **Know:** If $T \sim U(0, \theta)$, then $t_\alpha = \alpha\theta$. If $T_0 \sim U(0, \omega)$, then $T_x \sim U(0, \omega - x)$ has $\theta = \omega - x$. If $T \sim EXP(\mu)$ then $t_\alpha = \dfrac{-\log(1-\alpha)}{\mu}$.

169) **Know:** $Y = \overline{Y}_x = \dfrac{1-\overline{Z}_x}{\delta} = \dfrac{1-v^{T_x}}{\delta} = \dfrac{1-e^{-\delta T_x}}{\delta} = \overline{a}_{\overline{T_x|}} = g(T_x)$ is an increasing function of T_x. So the αth percentile of Y is $y_\alpha = g(t_\alpha) = \overline{a}_{\overline{t_\alpha|}} = \dfrac{1-e^{-\delta t_\alpha}}{\delta}$.

170) **Know:** $Z = \overline{Z}_x = v^{T_x} = e^{-\delta T_x} = g(T_x)$ is an decreasing function of T_x. So the αth percentile of Z is $z_\alpha = g(t_{1-\alpha}) = e^{-\delta t_{1-\alpha}}$.

171) $P(\overline{Y}_x \le y) = P\left(\frac{1-\overline{Z}_x}{\delta} \le y\right) = P\left(\frac{1-e^{-\delta T_x}}{\delta} \le y\right) = P(\overline{a}_{\overline{T_x|}} \le y) =$
$P\left(T_x \le \frac{\log(1-\delta y)}{-\delta}\right) = F_{T_x}\left(\frac{\log(1-\delta y)}{-\delta}\right)$ for $y < 1/\delta$.

172) $P(\overline{Z}_x \le z) = P(v^{t_x} \le z) = P(e^{-\delta T_x} \le z) = P\left(T_x > \frac{\log(z)}{-\delta}\right) = S_{T_x}\left(\frac{\log(z)}{-\delta}\right)$ where $0 \le z \le 1$.

If $T_x \sim EXP(\mu)$, then $P(\overline{Z}_x \le z) = z^{\mu/\delta}$ where $0 \le z \le 1$.

173) The 100 αth percentile premium π is the premium which results in a positive loss at issue with probability α. So want $\alpha = P(e^{-\delta T_x} > \pi \overline{a}_{\overline{T_x|}}) = P\left[e^{-\delta T_x} > \pi\left(\frac{1-e^{-\delta T_x}}{\delta}\right)\right] =$
$P\left(\overline{Z}_x > \frac{\pi}{\pi+\delta}\right) = P\left[T_x < \frac{-1}{\delta}\log\left(\frac{\pi}{\pi+\delta}\right)\right]$. This results in $\pi = \frac{\delta e^{-\delta t_\alpha}}{1-e^{-\delta t_\alpha}}$.

If the insurance benefit is K instead of 1, then $\pi = \frac{K\delta e^{-\delta t_\alpha}}{1-e^{-\delta t_\alpha}}$.

∮ 10.4

174) Assume the UDD assumption holds for all m decrements. Then the UDD assumption holds for the total decrement τ. So ${}_tq_x^{(j)} \approx (t)(q_x^{(j)})$ and ${}_tq_x^{(\tau)} \approx (t)(q_x^{(\tau)})$ for $j = 1, ..., m$ and $0 \le t \le 1$. Thus ${}_tp_x^{(k)} \approx 1 - (t)(q_x^{(k)})$ for $k = j, \tau$ and $0 \le t \le 1$.

175) If the UDD assumption holds for the jth decrement, then $q_x^{(j)} \approx {}_tp_x^{(\tau)}\mu_{x+t}^{(j)}$.

176) If the UDD assumption holds for the jth decrement and the total decrement τ and $0 \le t \le 1$, then i) $\mu_{x+t}^{(j)} \approx \frac{q_x^{(j)}}{{}_tp_x^{(\tau)}} \approx \frac{q_x^{(j)}}{1-(t)(q_x^{(\tau)})}$. For the single decrement table probabilities, ii) ${}_tp_x^{'(j)} \approx \left[1-(t)(q_x^{(\tau)})\right]^{q_x^{(j)}/q_x^{(\tau)}} \approx \left[{}_tp_x^{(\tau)}\right]^{q_x^{(j)}/q_x^{(\tau)}}$.

177) Can also assume that the m single decrement quantities satisfy the UDD assumption. So ${}_tq_x^{'(j)} \approx (t)(q_x^{'(j)})$ and ${}_tp_x^{'(j)}\mu_{x+t}^{(j)} \approx q_x^{'(j)}$ for $j = 1, ..., m$ and $0 \le t \le 1$.

178) In 177), assume $m = 2$. So $0 \le t \le 1$ and i) $q_x^{(1)} \approx q_x^{'(1)}\left(1 - \frac{q_x^{'(2)}}{2}\right)$
ii) ${}_tq_x^{(1)} \approx q_x^{'(1)}\left(t - \frac{t^2 q_x^{'(2)}}{2}\right)$, iii) ${}_{t|s}q_x^{(1)} \approx (s)(q_x^{'(1)})\left[1-\left(t+\frac{s}{2}\right)q_x^{'(2)}\right]$
where $0 < s + t \le 1$.

178) In 177), assume $m = 3$. Then $q_x^{(1)} \approx q_x^{'(1)}\left[1 - \frac{1}{2}(q_x^{'(2)} + q_x^{'(3)}) + \frac{1}{3}(q_x^{'(2)})(q_x^{'(3)})\right]$.

end ∮ 10.4 material

179) Recall $\delta = \log(1+i)$. Under the UDD assumption, i) $\overline{A}_x \approx \frac{i}{\delta}A_x$,
ii) ${}^2\overline{A}_x \approx \left(\frac{2i+i^2}{2\delta}\right)({}^2A_x)$, iii) $A_x^{(m)} \approx \frac{i}{i^{(m)}}A_x$,
iv) $\overline{a}_x = \frac{1-\overline{A}_x}{\delta} \approx \frac{\frac{i}{\delta}A_x}{\delta} \approx \frac{1}{\delta} - \frac{i}{\delta^2}A_x$.
The illustrative life table gives 1000 A_x and 1000 $({}^2A_x)$.

$\oint$ 10.4

180) Suppose the UDD assumption holds for all m decrements. Then the UDD assumption holds for the total decrement τ. Hence ${}_tq_x^{(k)} \approx (t)(q_x^{(k)})$ and ${}_tp_x^{(k)} \approx 1 - (t)(q_x^{(k)})$ for $k = \tau$ and $k = j = 1, ..., m$, and $0 \leq t \leq 1$.

181) If the UDD assumption holds for the jth decrement, then ${}_tp_x^{(\tau)}\mu_{x+t}^{(j)} \approx q_x^{(j)}$ for $0 \leq t \leq 1$.

182) If the UDD assumption holds for the jth decrement and the total decrement τ, let $0 \leq t \leq 1$. Then i) $\mu_{x+t}^{(j)} \approx \dfrac{q_x^{(j)}}{{}_tp_x^{(\tau)}} \approx \dfrac{q_x^{(j)}}{1 - (t)(q_x^{(\tau)})}$.

ii) For single decrement probabilities,

$${}_tp_x^{\prime(j)} \approx \left[\ {}_tp_x^{(\tau)}\right]^{q_x^{(j)}/q_x^{(\tau)}} \approx \left[1 - (t)(q_x^{(\tau)})\right]^{q_x^{(j)}/q_x^{(\tau)}}.$$

183) Can also assume that all m single decrement quantities satisfy the UDD assumption, so ${}_tq_x^{\prime(j)} \approx (t)(q_x^{\prime(j)})$ and ${}_tp_x^{\prime(j)}\mu_{x+t}^{(j)} \approx q_x^{\prime(j)}$ for $0 \leq t \leq 1$ and $j = 1, ..., m$. Let $0 \leq t \leq 1$. Recall that ${}_1q_x^{(1)} = q_x^{(1)}$ when $t = 1$.

i) If $m = 2$, then ${}_tq_x^{(1)} \approx q_x^{\prime(1)}\left(t - \dfrac{t^2\ q_x^{\prime(2)}}{2}\right)$.

ii) If $m = 3$, then $q_x^{(1)} \approx q_x^{\prime(1)}\left[1 - \dfrac{1}{2}(q_x^{\prime(2)} + q_x^{\prime(3)}) + \dfrac{1}{3}(q_x^{\prime(2)})(q_x^{\prime(3)})\right]$.

End $\oint$ 10.4 UDD approximation material.

184) Recall $\delta = \log(1+i)$. Under the UDD assumption, $\overline{A}_x \approx \dfrac{i}{\delta}A_x$,

$${}^2\overline{A}_x \approx \left(\frac{2i + i^2}{2\delta}\right)(\ {}^2A_x), \quad \overline{A}_x^{(m)} \approx \frac{i}{i^{(m)}}A_x, \quad \text{and} \quad \overline{a}_x = \frac{1 - \overline{A}_x}{\delta} \approx \frac{1}{\delta} - \frac{i}{\delta^2}A_x.$$

185) Insurance formulas for APV $\overline{A}$ if T_x is Exponential or DeMoivre.

insurance	$T_x \sim EXP(\mu)$	$T_x \sim U(0, \omega - x)$, $n < \omega - x$
whole life $\overline{A}_x$	$\dfrac{\mu}{\mu + \delta}$	$\dfrac{1}{\omega - x}\ \dfrac{1 - e^{-\delta(\omega - x)}}{\delta} = \dfrac{\overline{a}_{\overline{\omega - x}\rvert}}{\omega - x}$
n year term $\overline{A}^1_{x:\overline{n}\rvert}$	$\dfrac{\mu}{\mu + \delta}\left[1 - e^{-n(\mu+\delta)}\right]$	$\dfrac{1}{\omega - x}\ \dfrac{1 - e^{-\delta n}}{\delta} = \dfrac{\overline{a}_{\overline{n}\rvert}}{\omega - x}$
n year deferred life ${}_{n\rvert}\overline{A}_x$	$\dfrac{\mu}{\mu + \delta}\ e^{-n(\mu+\delta)}$	$\dfrac{e^{-\delta n}}{\omega - x}\ \dfrac{1 - e^{-\delta(\omega - x - n)}}{\delta} = \dfrac{e^{-\delta n}}{\omega - x}\ \overline{a}_{\overline{\omega - x - n}\rvert}$
n year pure endowment $A_{\underset{x:\overline{n}\rvert}{1}} = {}_nE_x$	$e^{-n(\mu+\delta)}$	$\dfrac{e^{-\delta n}}{\omega - x}\ (\omega - x - n)$

For ${}^2\overline{A}$, replace δ by 2 δ in the above table. Recall that the pure endowment is both continuous and discrete, and the discrete insurance notation is used.

$\oint$ 9.4

186) A *revisionary annuity* is for two lives (x) and (y). a) If the beneficiary is (y), then provided that (y) survives (x), after (x) dies, (y) receives an annuity until (y) dies. If (y) dies first, then the insurance contract ends and no benefits are paid. b) Similarly, if the beneficiary is (x), then provided that (x) survives (y), after (y) dies, (x) receives an annuity until (x) dies. If (x) dies first, then the insurance contract ends and no benefits are paid. Suppose the insurance is as in a). i) discrete whole life $a_{x|y} = a_y - a_{xy}$. ii) discrete n year term $a_{x|y:\overline{n}|} = a_{y:\overline{n}|} - a_{xy:\overline{n}|}$. iii) continuous whole life $\overline{a}_{x|y} = \overline{a}_y - \overline{a}_{xy}$. iv) continuous n year term term $\overline{a}_{x|y:\overline{n}|} = \overline{a}_{y:\overline{n}|} - \overline{a}_{xy:\overline{n}|}$.

187) Premiums are paid until one of (x) or (y) dies (failure of joint life status). Assume the equivalence principle is used. Consider insuarance a) in 186). i) If the fully discrete whole life insurance is funded by annual premiums, then $P(a_{x|y}) = \dfrac{a_{x|y}}{\ddot{a}_{xy}} = \dfrac{a_y - a_{xy}}{\ddot{a}_{xy}}$. ii) If the fully continuous whole life insurance is funded by a continuously paid premium, then $\overline{P}(\overline{a}_{x|y}) = \dfrac{\overline{a}_{x|y}}{\overline{a}_{xy}} = \dfrac{\overline{a}_y - \overline{a}_{xy}}{\overline{a}_{xy}}$.

188) Referring to 187), if both (x) and (y) are alive, then ${}_tV(a_{x|y}) = a_{x+t|y+t} - P(a_{x|y})\ddot{a}_{x+t:y+t}$ and ${}_t\overline{V}(\overline{a}_{x|y}) = \overline{a}_{x+t|y+t} - \overline{P}(\overline{a}_{x|y})\overline{a}_{x+t:y+t}$. If only (y) is alive then ${}_tV(a_{x|y}) = a_{y+t}$ and ${}_t\overline{V}(\overline{a}_{x|y}) = \overline{a}_{y+t}$. If only (x) is alive then the contract is expired and the benefit reserves ${}_tV(a_{x|y}) = 0$ and ${}_t\overline{V}(\overline{a}_{x|y}) = 0$.

189) For revisionary insurance b), use formulas like $\overline{a}_{y|x} = \overline{a}_x - \overline{a}_{xy}$.

Markov Chains: Insurance, Annuities, Premiums, Reserves

190) For discrete insurance Z that pays benefit b_k if $\lceil T_x \rceil = k$ at time k with probability $p_k = P(\lceil T_x \rceil = k) = {}_{k-1}q_x$, the possible values of Z are $b_k v^k$ which occur with probability p_k. Hence $APV = E(Z) = \sum_{k=1}^m b_k v^k p_k$ where $m = \infty$ is possible. Note that the summand is the triple product of i) the benefit b_k, ii) the discount factor v^k, and iii) the probability p_k. The APV of a set of cashflows using a Markov chain is similar. The APV is the sum of a triple product of i) the cashflow c_k, ii) the discount factor v^k, and iii) the probability p_k that the cashflow occurs, computed using the Markov chain. So for an insurance Z, the possible values are of Z are $c_k v^k$ which occur with probability p_k.

Lots of variants are possible. For example there could be states i) alive, ii) death from accident, and iii) death from non-accident. Could have benefit $b_{k,1}$ at time k if death occurred due to accident with probability $p_{k,1}$, and benefit $b_{k,2}$ at time k if death occurred due to non-accident with probability $p_{k,2}$, where the probabilities are determined using the Markov chain. Then APV $= \sum_k (b_{k,1} v^k p_{k,1} + b_{k,2} v^k p_{k,2})$, but the principle is the same, each summand is the triple product: (cashflow at time k)$(v^k)(p_k)$ where p_k is the probability that the cashflow is made.

191) An annuity with Markov chains is similar. There will be consecutive possible cash flows, say $C_1, ..., C_m$ at times 1, ..., m which are paid with probability p_k determined from the Markov chain. Then the APV $= \sum_{k=1}^m C_k v^k p_k$.

192) The premium is determined from the equivalence principle. Let the APV(benefits) be computed as in 190) or 191). The annuity due of 1 for $J+1$ periods has possible values $v^0, v^1, ..., v^J$. Assume these have probabilities p_i where $p_0 = 1$ since the first premium of 1 is certain to be paid. Then the APV(annuity due of 1) is $\sum_{i=0}^J v^i p_i$, and the premium is $P =$ APV(benefits)/(APV(annuity due of 1). Time diagrams are useful.