Math 403 Exam 2 is Wed. Oct. 18. **You are allowed 10 sheets of notes and a calculator.** The exam emphasis is HW4-6, and Q4-6. Numbers refer to types of problems on exam.

Bring Exam 1 review pages 1–2 to all exams. Points 7), 8), 9), 10), 25), 29), 36c), 37), 43), 45)–51) may also be useful.

52) If $E(N)$ is not much less than 100, then the normal approximation is $P(S_N \le x) = P\left(Z \le \frac{x - E(S_N)}{\sqrt{V(S_N)}}\right)$. See 49) and note that S is often **used instead** of S_N. $VaR_p(S_n) = \pi_p \approx E(S_N) + z_p SD(S_N)$.

53) For the $(a, b, 0)$ class, $\frac{p_k}{p_{k-1}} = a + \frac{b}{k}$ for $k = 1, 2, ...$ Hence $\frac{k\ p_k}{p_{k-1}} = a\ k + b$ for $k = 1, 2, ...$ The Poisson RV N has slope $a = 0$ and $E(N) = V(N)$. The bin RV N has slope $a < 0$ and $E(N) > V(N)$. The NB RV N has slope $a > 0$ and $E(N) < V(N)$.

dist	a	b	p_0
Poisson(λ)	0	λ	$e^{-\lambda}$
bin(q, m)	$\frac{-q}{1-q}$	$(m+1)\frac{q}{1-q}$	$(1-q)^m$
NB(β, r)	$\frac{\beta}{1+\beta}$	$(r-1)\frac{\beta}{1+\beta}$	$(1+\beta)^{-r}$
geom(β)	$\frac{\beta}{1+\beta}$	0	$(1+\beta)^{-1}$

54) Let p_k be the pmf of the Poisson, binomial or NB RV. Denote the pmf of a zero truncated ZT Poisson, ZT binomial, or ZT NB RV N by p_k^T. Denote the pmf of a zero modified ZM Poisson, ZM binomial, or ZM ETNB RV N by p_k^M. Then $p_k^T = \frac{p_k}{1 - p_0}$ and $p_0^T = 0$, while $p_k^M = \frac{1 - p_0^M}{1 - p_0} p_k$ for $k = 1, 2, ...$, and $p_0^M \in [0, 1]$. Sometimes p_0^M is known and sometimes treated as a parameter to be estimated.

55) **Know** Suppose $N|\lambda \sim$ Poisson(λ) and $\lambda \sim G(\alpha, \theta)$. Then $N \sim NB(\beta = \theta, r = \alpha)$. $E(N) = E(\lambda) = \alpha\theta$ and $V(N) = E(N) + V(\lambda) = \alpha\theta + \alpha\theta^2 = \alpha\theta(1 + \theta)$. Often X is used instead of N and Λ instead of λ. Often told the mean and variance of the NB RV N, then find α and θ or the variance of λ. Note that λ is a RV, not a parameter.

56) Suppose $z_p = \pi_p$ for the N(0,1) distribution: $P(Z \le z_p) = p$.

X	$VaR_p(X) = \pi_p$	$TVaR_P(X)$
$N(\mu, \sigma^2)$	$\mu + \sigma z_p$	$\mu + \frac{\sigma\phi(z_p)}{1-p}$
$LN(\mu, \sigma)$	$\exp(\mu + \sigma z_p)$	$\frac{E(X)\Phi(\sigma - z_p)}{1-p}$

If $X \sim LN(\mu, \sigma)$, then X has a lognormal distribution and $\ln(X) \sim N(\mu, \sigma^2)$. $\Phi(x) = P(Z \le x)$ is the cdf of $Z \sim N(0, 1)$ (see normal tables), and $\phi(x) = f_Z(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$ is the N(0,1) pdf. For $a > 0$, $aX \sim LN(\mu + \ln(a), \sigma)$.

57) A payment per loss has 0 as a possibility where there is a loss without a payment due to a deductible. The left censored and shifted RV = per loss RV $Y^L = (X - d)_+$. So $Y^L = 0$ for $X < d$, and $Y_L = X - d$ for $X > d$. Recall $(X - d)_+ = \max(X - d, 0)$.

58) For payment per payment, the excess loss RV = per payment RV Y^P is undefined when there is no payment, ie for $X < d$. $Y^P = Y^L | Y^L > 0 = (X - d)|X > d = X - d$ for $X > d$. See 9).

59) The RVs in 57) and 58) are for an **ordinary deductible**. A *franchise deductible* pays $Y^L = 0$ if $X \leq d$ but pays $Y^L = X$ if $X > d$. So the franchise deductible pays the full amount X if $X > d$. For a franchise deductible, Y^P is undefined for $X < d$, and $Y^P = X$ for $X > d$. **Assume a deductible is an ordinary deductible** unless stated otherwise.

60) i) If $X \sim EXP(\theta)$, then $Y^P \sim EXP(\theta)$ and $e_X(d) = \theta$.
ii) If $X \sim U(0, \theta)$ and $d < \theta$, then $Y^P \sim U(0, \theta - d)$ and $e_X(d) = (\theta - d)/2$.
iii) If $X \sim$ (two parameter) Pareto (α, θ) then $Y^P \sim$ (two parameter) Pareto $(\alpha, \theta + d)$, and for $\alpha > 1$, $e_X(d) = \dfrac{\theta + d}{\alpha - 1}$.
iv) If $X \sim$ single parameter Pareto (α, θ) and $\alpha > 1$, then $e_X(d) = \dfrac{d}{\alpha - 1}$ for $d \geq \theta$, and $e_X(d) = \dfrac{\alpha(\theta - d) + d}{\alpha - 1}$ for $d \leq \theta$. If $d \geq \theta$, then $Y^P \sim$ (two parameter) Pareto (α, d).

61) The **mean excess loss** $= E(Y^P) = e_X(d) = \dfrac{E(Y^L)}{S_X(d)} = \dfrac{E[(X-d)_+]}{S_X(d)} = \dfrac{\int_d^\infty S_X(x)dx}{S_X(d)} = \dfrac{\int_d^\infty (x-d) f_X(x)dx}{S_X(d)} = \dfrac{E[(X) - E[X \wedge d]}{S_X(d)}$. Note that $E(Y^L)$ is given in the numerator. Tables give $E(X)$, $E(X \wedge x)$, $F_X(x) = 1 - S_X(x)$, $VaR_p(X)$ and $TVaR_p(X)$. Recall that $TVaR_p(X) = VaR_p(X) + e_X(\pi_p) = \pi_p + e_X(\pi_p)$.

62) Let X be a loss RV. Then for $y > 0$,
i) $f_{Y^P}(y) = \dfrac{f_X(y+d)}{S_X(d)}$, ii) $S_{Y^P}(y) = \dfrac{S_X(y+d)}{S_X(d)}$, iii) $F_{Y^P}(y) = \dfrac{F_X(y+d) - F_X(d)}{S_X(d)}$, and iv) $h_{Y^P}(y) = \dfrac{f_X(y+d)}{S_X(y+d)} = h_X(y+d)$. Since Y^L is a mixture of a point mass at 0 and Y^P, the pdf of Y^L does not exist. v) $F_{Y^L}(y) = F_X(y+d)$, and $S_{Y^L}(y) = S_X(y+d)$. Y^P is a continuous RV, so the formulas from 1) still hold.

63) Let $Y^L(O)$ and $Y^P(O)$ be the loss RV and payment RV for an ordinary deductible (d), and let $Y^L(F)$ and $Y^P(F)$ be the loss RV and payment RV for a franchise deductible (d). Then $E[Y^L(F)] = E[Y^L(O)] + dS_X(d) = E(X) - E(X \wedge d) + dS_X(d)$, and $E[Y^P(F)] = E[Y^P(O)] + d = \dfrac{E(Y^L(F)]}{S_X(d)}$. This expectation makes sense because the policy with a franchise deductible pays d more than that of a policy with an ordinary deductible when $X > d$. Usually the F and O are suppressed. $E[Y^L(F)] = \int_d^\infty x f(x)dx$.

64) For a franchise deductible, let X be a loss RV. Then for $y > d$,
i) $f_{Y^P}(y) = \dfrac{f_X(y)}{S_X(d)}$, ii) $S_{Y^P}(y) = \dfrac{S_X(y)}{S_X(d)}$, iii) $F_{Y^P}(y) = \dfrac{F_X(y) - F_X(d)}{S_X(d)}$, and iv) $h_{Y^P}(y) = h_X(y)$. For $0 < y < d$, $f_{Y^P}(y) = 0$, $S_{Y^P}(y) = 1$, $F_{Y^P}(y) = 0$ and $h_{Y^P}(y) = 0$. The pdf of Y^L does not exist. v) $F_{Y^L}(y) = F_X(y)$ for $y > d$, and $F_{Y^L}(y) = F_X(d)$ for $0 < y < d$.

vi) $S_{Y^L}(y) = S_X(y)$ for $y > d$, and $S_{Y^L}(y) = S_X(d)$ for $0 < y < d$.
vii) $h_{Y^L}(y) = h_X(y)$ for $y > d$, and $h_{Y^L}(y) = h_X(y)$ for $0 < y < d$.

65) The **loss elimination ratio** LER $= \dfrac{E[X \wedge d]}{E(X)}$ if $E(X)$ exists. Note that $E(Y^L) = E[(X-d)_+] = E(X) - E[X \wedge d]$. So $E[X \wedge d] = E(X) - E[(X-d)_+] = E(X) - E(Y^L)$.

66) Let the annual inflation rate be r where often you are told the uniform inflation rate is $1+r$ (usually $0 < r < 1$). After inflation, the new loss RV $Y = (1+r)X$.

67) **Useful**: Nearly all of the continuous distributions in Appendix A with parameter θ are scale families with scale parameter θ if any other parameters $\boldsymbol{\tau}$ are fixed, written $X \sim SF(\theta|\boldsymbol{\tau})$. Let $a > 0$ where often $a = 1+r$. Then $Y = aX \sim SF(a\theta|\boldsymbol{\tau})$. The inverse Gaussian distribution is an exception. If $X \sim LN(\mu, \sigma)$, then $Y = aX \sim LN(\mu + \ln(a), \sigma)$.

68) If $Y = (1+r)X$ for loss RV X, then
i) $E[Y \wedge d] = (1+r)E\left[X \wedge \dfrac{d}{1+r}\right]$,
ii) $E(Y) = (1+r)E(X)$, iii) $F_Y(d) = F_X\left(\dfrac{d}{1+r}\right)$, iv) $S_Y(d) = S_X\left(\dfrac{d}{1+r}\right)$.

69) For an ordinary deductible of d, **after uniform inflation** of $1+r$, **method I):**
i) $E(Y^L) = (1+r)\left[E(X) - E\left[X \wedge \dfrac{d}{1+r}\right]\right]$, ii) $E(Y^P) = \dfrac{E(Y^L)}{S_X\left(\frac{d}{1+r}\right)}$.

Method II): If $X \sim SF(\theta|\boldsymbol{\tau})$, then the $X_{new} = (1+r)X$ satisfies $X_{new} \sim SF((1+r)\theta|\boldsymbol{\tau})$. Use this modified distribution and formulas 61). See 67).

70) If $W \sim AN(\mu, \sigma^2)$, eg $W = S_N$, then the normal approximation $VaR_p(W) = \pi_p \approx \mu + z_p\sigma$.

71) For a policy limit, the limited loss RV $W = X \wedge u = \min(X, u)$, and $E(X \wedge u)$ is the limited expected value. $F_{X\wedge u}(y) = F_X(y)$ for $y < u$ and $F_{X\wedge u}(y) = 1$ for $y \geq u$.

72) Policy limit and insurance: Let $Y = (1+r)X = X_{new}$.
Method 1: $E(Y \wedge u) = E(X_{new} \wedge u) = (1+r)E\left[X \wedge \dfrac{u}{1+r}\right]$.

Method 2: If $X \sim SF(\theta|\boldsymbol{\tau})$, then $X_{new} = (1+r)X$ satisfies $X_{new} \sim SF((1+r)\theta|\boldsymbol{\tau})$. Get $E(X_{new} \wedge u)$ and the table formulas for the modified distribution.

73) Let X be a loss RV. For a coinsurance policy, the insurance company pays αX of the loss for some $\alpha \in (0, 1]$. For coinsurance with a deductible, the insurance company pays $\alpha(X-d)_+$.

74) **Policy limit and a deductible:** If there is a deductible d and a policy limit = maximum payment of $u - d$, then the "maximum covered loss" $u = u - d + d$. (A loss $X > u$ is not fully covered in that the policy will only pay $u - d$ instead of $X - d$.) The the per loss RV

$$Y^L = X \wedge u - X \wedge d = \begin{cases} 0, & X < d \\ X - d, & d \leq X < u \\ u - d & X \geq u. \end{cases}$$

Then $E(Y_L) = E(X \wedge u) - E(X \wedge d)$ and $E(Y_P) = \dfrac{E(Y^L)}{S_X(d)}$ where $Y^P = Y^L | X > d$.

75) **Insurance with an ordinary deductible** d**, policy limit** $u-d$**, coinsurance** α**, and inflation** r: The the per loss RV

$$Y^L = \begin{cases} 0, & X < \frac{d}{1+r} \\ \alpha[(1+r)X - d], & \frac{d}{1+r} \le X < \frac{u}{1+r} \\ \alpha(u-d) & X \ge \frac{u}{1+r}. \end{cases}$$

Then $E(Y_L) = \alpha(1+r)\left[E\left(X \wedge \dfrac{u}{1+r}\right) - E\left(X \wedge \dfrac{d}{1+r}\right)\right]$ and $E(Y_P) = \dfrac{E(Y^L)}{S_X\left(\frac{d}{1+r}\right)}$.

Note that $\alpha = 1$ for no coinsurance, $r = 0$ for no inflation, $d = 0$ for no deductible, and take $u = \infty$ if there is no policy limit. So $E(X \wedge \infty] = E(X)$ if there is no policy limit.

76) $(1+r)X \wedge d = (1+r)\left(X \wedge \dfrac{d}{1+r}\right)$ Since $E[(1+r)X] = (1+r)E[X]$, after inflation, $LER(d) = \dfrac{E\left[X \wedge \frac{d}{1+r}\right]}{E[X]}$. If $X \sim SF(\theta|\boldsymbol{\tau})$ then $X_{new} \sim SF((1+r)\theta|\boldsymbol{\tau})$ gives another method to find $LER(d)$ after inflation.

77) **Useful**: i) If $X \sim EXP(\theta)$, then $LER(d) = 1 - e^{-d/\theta}$.

ii) If $X \sim$ (two parameter) Pareto $(\alpha > 1, \theta)$ then $LER(d) = 1 - \left(\dfrac{\theta}{d+\theta}\right)^{\alpha-1}$,

iii) If $X \sim$ single parameter Pareto $(\alpha > 1, \theta)$ and $d > \theta$, then $LER(d) = 1 - \dfrac{(\theta/d)^{\alpha-1}}{\alpha}$.

78) A mixed frequency distribution has one or more parameters random. So a mixed Poisson distribution has λ a RV. Hence if N has a mixed Poisson distribution and $\lambda \sim G(\alpha, \theta)$, then $N \sim NB(\beta = \theta, r = \alpha)$. Formulas from 43) will often be useful.