Exam 1 on Wed. Feb. 9 will cover sections 1.1–1.6 and 2.1–2.5.

Memorization tip: On the left hand side of a piece of paper, write key words like "$N(\mu, \sigma^2)$" or "kernel method." On the right hand side (RHS) write what you want to memorize for each key word. When you think you have your list memorized, cover the RHS with a piece of paper and try to write down what you want to memorize. Repeat until you can write down everything correctly.

No notes will be allowed, but bring a calculator. You should memorize the pmf or pdf f, $E(Y)$ and V(Y) for the following RVs: 1) beta(δ, ν),
2) Bernoulli(ρ) = bin$(k = 1, \rho)$, 3) binomial(k, ρ), 4) Cauchy(μ, σ),
5) chi-square(p) = gamma$(\nu = p/2, \lambda = 2)$, 6) exponential(λ) = gamma$(\nu = 1, \lambda)$,
7) gamma(ν, λ), 8) $N(\mu, \sigma^2)$, 9) Poisson(θ), and 10) uniform(θ_1, θ_1).

You should memorize the mgf of the binomial, χ^2_p, exponential, gamma, normal and Poisson distributions. You should memorize the cdf of the exponential and of the normal distribution $\Phi(\frac{y - \mu}{\sigma})$.

The terms sample space S, events, disjoint, partition, probability function, conditional probability, mutually independent events, random variable, cdf, properties of cdf, continuous RV, discrete RV, identically distributed, pmf, pdf, properties of pmfs and pdfs are important.

Types of problems:

1) Find $E[g(Y)]$, especially $E(Y) = m'(0)$, $E(Y^2) = m''(0)$, $V(Y) = E(Y^2) - [E(Y)]^2$, and the mgf $m(t) = m_Y(t) = E[e^{tY}]$. You may need to compute moments using the mgf. See Q1 2,3, HW1.

2) Recall that $f_Y(y|\boldsymbol{\theta}) = c(\boldsymbol{\theta})k(y|\boldsymbol{\theta})$ where $k(y|\boldsymbol{\theta})$ is the **kernel** of f_Y.
Thus $\int_{-\infty}^{\infty} k(y|\boldsymbol{\theta})dy = 1/c(\boldsymbol{\theta})$. Often $E[g(Y)] =$

$$a\ c(\boldsymbol{\theta}) \int_{-\infty}^{\infty} k(y|\boldsymbol{\tau})dy = a\ c(\boldsymbol{\theta}) \frac{1}{c(\boldsymbol{\tau})} \int_{-\infty}^{\infty} c(\boldsymbol{\tau})k(y|\boldsymbol{\tau})dy = \frac{a\ c(\boldsymbol{\theta})}{c(\boldsymbol{\tau})}$$

for some constant a. Use the kernel method to find $E[g(Y)]$. Complete the square for the Gaussian kernel. The kernel method is especially useful for finding $E[Y^n]$. Replace the integral by a sum for a discrete distribution. See Q1 1, Q2 1, HW1, HW2 4.

3) Know how to find $E[g(X)]$ if X follows a mixture distribution. If the cdf of X is $F_X(x) = (1 - \epsilon)F_Z(x) + \epsilon F_W(x)$ where $0 \le \epsilon \le 1$ and F_Z and F_W are cdfs, then $E[g(X)] = (1 - \epsilon)E[g(Z)] + \epsilon E[g(W)]$. In particular, $E(X^2) = (1 - \epsilon)E[Z^2] + \epsilon E[W^2] = (1 - \epsilon)[V(Z) + (E[Z])^2] + \epsilon[V(W) + (E[W])^2]$. See Q2 3, HW2 3.

4) Know that the conditional pdf $f(y|x)$ is a function of y but the conditional expectation $E(Y|X = x)$ is a function of x.

5) Know how to find $E[Y] = E[E(Y|X)]$ and $V(Y) = E[V(Y|X)] + V[E(Y|X)]$. The formula $V(W) = E(W^2) - [E(W)]^2$ is often useful for this type of problem where W has the same distribution as $Y|X = x$. See Q2 2, HW3 1.

6) Be able to find $EaX + b$ when a and b are constants.

7) Y_1 and Y_2 are dependent if the support $\mathcal{Y} = \{(y_1, y_2)|f(y_1, y_2) > 0\}$ is not a cross product. See Q3 4, HW3 3.

8) If the support is a cross product, then Y_1 and Y_2 are independent iff $f(y_1, y_2) = h_1(y_1)h_2(y_2)$ for all $(y_1, y_2) \in \mathcal{Y}$ where $h_i(y_i)$ is a positive function of y_i alone. If no such factorization exists, then Y_1 and Y_2 are dependent. See HW3 2d.

9) If $Y_1, ..., Y_n$ are independent, then the functions $h_1(Y_1), ..., h_n(Y_n)$ are independent.

10) Given $f(y_1, y_2)$, find $E[h(Y_i)]$ by finding the marginal pdf or pmf $f_{Y_i}(y_i)$ and using the marginal distribution in the expectation. **Formulas for pmfs are like those for pdfs, but replace the integrals by sums.**

11) Find the pmf of $Y = t(X)$ and the sample space $\mathcal{Y}$ given the pmf of X. See Q3 and HW3 4.

12) Find the pdf of $Y = t(X)$ and the sample space $\mathcal{Y}$ given the pdf of X. For increasing or decreasing t,

$$f_Y(y) = f_X(t^{-1}(y)) \left| \frac{dt^{-1}(y)}{dy} \right|$$

for $y \in \mathcal{Y}$. See Q3 and HW3 5,6.

13) Find the joint pdf of $Y_1 = t_1(X_1, X_2)$ and $Y_2 = t_2(X_1, X_2)$: $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(t_1^{-1}(y_1, y_2), t_2^{-1}(y_1, y_2))|J|$. Finding the support $\mathcal{Y}$ is crucial. Using indicator functions can help. Know that $\prod_{j=1}^k I_{A_j}(\boldsymbol{y}) = I_{\cap_{j=1}^k A_j}(\boldsymbol{y})$. Be able to get a marginal distribution, eg $f_{Y_1}(y_1)$ given the joint. Also know how to show that Y_1 and Y_2 are independent. You should be able to sketch $\mathcal{Y}$ if t_1 and t_2 are linear. The Jacobian of the bivariate transformation is

$$J = \det \begin{bmatrix} \frac{\partial t_1^{-1}}{\partial y_1} & \frac{\partial t_1^{-1}}{\partial y_2} \\ \frac{\partial t_2^{-1}}{\partial y_1} & \frac{\partial t_2^{-1}}{\partial y_2} \end{bmatrix},$$

and $|J|$ is the absolute value of the determinant J. Recall that

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{ad} - \text{bc}.$$

To find $t_i^{-1}(y_1, y_2)$, use $y_i = t_i(x_1, x_2)$ and solve for x_1 and x_2 where $i = 1, 2$. See Q3 and HW3 7.

14) Typical Math 483 calculations such as finding c so $\int ck(x|\theta) = 1$, finding conditional and marginal distributions from joint distributions, determining whether X and Y are independent given $f_{X,Y}(x, y)$, finding $Cov(X, Y) = E(XY) - E(X)E(Y)$ and finding $P[(X, Y) \in A]$ could appear on the exam. You should be able to find the cdf F from f and able to find f from the cdf F. Q3, HW3 2,3

MEMORIZE!

1) beta(δ, ν)

$$f(y) = \frac{\Gamma(\delta+\nu)}{\Gamma(\delta)\Gamma(\nu)} y^{\delta-1}(1-y)^{\nu-1} \text{ where } \delta > 0,\ \nu > 0 \text{ and } 0 \le \text{y} \le 1.$$

$$E(Y) = \frac{\delta}{\delta+\nu}.$$

$$\text{VAR}(Y) = \frac{\delta\nu}{(\delta+\nu)^2(\delta+\nu+1)}.$$

2) Bernoulli(ρ) = binomial$(k = 1, \rho)$ $\quad f(y) = \rho^y(1-\rho)^{1-y}$ for $y = 0, 1$.
$E(Y) = \rho$.
VAR$(Y) = \rho(1-\rho)$.

$$m(t) = [(1-\rho) + \rho e^t].$$

3) binomial(k, ρ)

$$f(y) = \binom{k}{y} \rho^y (1-\rho)^{k-y} \text{ for } \text{y} = 0, 1, \ldots, \text{k} \text{ where } 0 < \rho < 1.$$

$E(Y) = k\rho$.
VAR$(Y) = k\rho(1-\rho)$.

$$m(t) = [(1-\rho) + \rho e^t]^k.$$

4) Cauchy(μ, σ)

$$f(y) = \frac{1}{\pi\sigma[1 + (\frac{y-\mu}{\sigma})^2]}$$

where y and μ are real numbers and $\sigma > 0$.
$E(Y) = \infty =$ VAR(Y).

5) chi-square(p) = gamma$(\nu = p/2, \lambda = 2)$

$$f(y) = \frac{y^{\frac{p}{2}-1} e^{-\frac{y}{2}}}{2^{\frac{p}{2}} \Gamma(\frac{p}{2})}$$

for $y > 0$.
$E(Y) = p$.
VAR$(Y) = 2p$.

$$m(t) = \left(\frac{1}{1-2t}\right)^{p/2} = (1-2t)^{-p/2} \text{ for } \text{t} < 1/2.$$

6) exponential(λ)= gamma($\nu = 1, \lambda$)

$$f(y) = \frac{1}{\lambda} \exp\left(-\frac{y}{\lambda}\right) \; I(y \geq 0) \quad \text{where} \quad \lambda > 0.$$

$E(Y) = \lambda$,
VAR$(Y) = \lambda^2$.

$$m(t) = 1/(1 - \lambda t) \quad \text{for} \quad \text{t} < 1/\lambda.$$

$$F(y) = 1 - \exp(-y/\lambda), \; y \geq 0.$$

7) gamma(ν, λ)

$$f(y) = \frac{y^{\nu-1} e^{-y/\lambda}}{\lambda^{\nu} \Gamma(\nu)} \quad \text{where} \quad \nu, \; \lambda \quad \text{and} \quad \text{y} \quad \text{are} \quad \text{positive.}$$

$E(Y) = \nu\lambda$.
VAR$(Y) = \nu\lambda^2$.

$$m(t) = \left(\frac{1}{1 - \lambda t}\right)^{\nu} \quad \text{for} \quad \text{t} < 1/\lambda.$$

8) $N(\mu, \sigma^2)$

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y-\mu)^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and μ and y are real.
$E(Y) = \mu$. VAR$(Y) = \sigma^2$.

$$m(t) = \exp(t\mu + t^2\sigma^2/2).$$

$$F(y) = \Phi\left(\frac{y - \mu}{\sigma}\right).$$

9) Poisson(θ)

$$f(y) = \frac{e^{-\theta} \theta^y}{y!} \quad \text{for} \quad \text{y} = 0, 1, \ldots, \quad \text{where} \quad \theta > 0.$$

$E(Y) = \theta =$ VAR(Y).

$$m(t) = \exp(\theta(e^t - 1)).$$

10) uniform(θ_1, θ_2)

$$f(y) = \frac{1}{\theta_2 - \theta_1} I(\theta_1 \leq y \leq \theta_2).$$

$F(y) = (y - \theta_1)/(\theta_2 - \theta_1)$ for $\theta_1 \leq y \leq \theta_2$.
$E(Y) = (\theta_1 + \theta_2)/2$.
VAR$(Y) = (\theta_2 - \theta_1)^2/12$.