Math 401 review

Review of Ch 2-5 Refer to Exam 1 review, 1-12.

Ch 2-3:

1] Know $X \sim$ Exponential (μ)

$f(x) = \mu e^{-\mu x}, \quad x > 0$

$E(X) = \frac{1}{\mu}, \quad V(X) = \frac{1}{\mu^2}$

$F(x) = 1 - e^{-\mu x}, \quad x > 0. \quad {}_tP_x = S(x) = e^{-\mu x}, \quad x > 0.$

In this class, usually $\mu < 1$, $E(X) > 1$.

2) Know ($X \sim U(\theta_1, \theta_2)$ especially the)

DeMoivre(θ) $= U(0, \theta)$ distribution.

$f(x) = \frac{1}{\theta}, \quad 0 \le x \le \theta.$

$E(X) = \frac{\theta}{2}, \quad V(X) = \frac{\theta^2}{12}$

$F(x) = \frac{x}{\theta}, \quad 0 \le x \le \theta$

3) Usually $X \ge 0$ is nonnegative and interest is in the survival function $S(x) = 1 - F(x)$.

4) cdf $F(x) = P(X \le x)$, $S(x) = P(X > x)$,

pdf $f(x) = F'(x)$, force of mortality $\mu(x)$ = hazard rate function, $E(X) = \mathring{e}_0$

5) (x) denotes a person alive at age x. 1.5

6) Let $X = T_0$ where $T_x = T(x)$ is time until failure (death) for a person alive at age x. $T_0 = x + T_x$ given $T_0 > x$.

7)

Let $G_x = G_{T_x}$

G is T, S, F, μ or f. If there is no subscript x, then $G = G_0$. Let $t > 0$.

i) $P[T_x > t] = {}_tp_x = S_x(t) = \dfrac{S_0(x+t)}{S_0(x)} = \exp\left(-\int_x^{x+t} \mu_y \, dy\right) = \exp\left(-\int_0^t \mu_{x+w} dw\right)$.

ii) ${}_tq_x = F_x(t) = 1 - {}_tp_x = 1 - \dfrac{S_0(x+t)}{S_0(x)} = P(T_x \le t)$

$= P(T_0 \le x+t \mid T_0 > x)$.

iii) ${}_tp_x \, \mu_{x+t} = f_x(t) = \dfrac{f_0(x+t)}{S_0(x)} = \dfrac{d}{dt} F_x(t) = \dfrac{-d}{dt} S_x(t)$

iv) $\mu_{x+t} = \mu_x(t) = \mu_0(x+t) = \dfrac{f_0(x+t)}{S_0(x+t)} = \dfrac{f_x(t)}{S_x(t)}$

* These quantities are nonnegative ≥ 0.
LHS = actuarial notation

8)* If $t=1$, the subscript is usually suppressed. So $p_x = {}_1p_x$ and $q_x = {}_1q_x$.

9) The complete expectation of life at age x = expected future lifetime at age x =

$$\mathring{e}_x = E(T_x) = \int_0^\infty t f_x(t)\,dt = \int_0^\infty {}_tp_x\,dt = \int_0^\infty S_x(t)\,dt.$$

Note that $\mathring{e}_0 = E(T_0)$.

10) $V(T_x) = E(T_x^2) - [E(T_x)]^2$ where

$$E(T_x^2) = \int_0^\infty t^2 f_x(t)\,dt = 2\int_0^\infty t\ {}_tp_x\,dt$$

$= 2\int_0^\infty t\, S_x(t)\,dt$. Use $x=0$ to find $E(T_0^2)$.

11) If $T_0 \sim U(0,\omega)$, then T_x has a DeMoivre$(\omega - x)$ distribution: $T_x \sim U(0, \omega - x)$, with support $0 < t < \omega - x$. For such a t,

$$S_x(t) = {}_tp_x = \frac{\omega - x - t}{\omega - x} = 1 - \frac{t}{\omega - x},$$

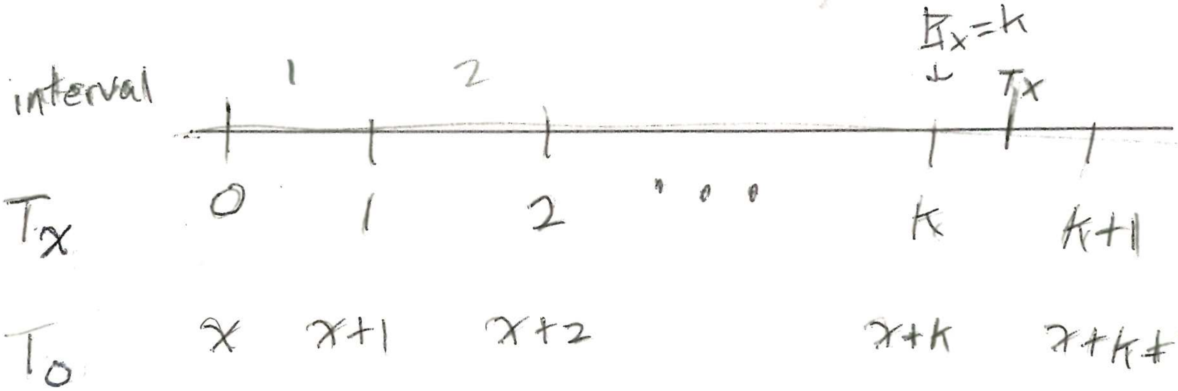
$$\mu_x(t) = \mu_{x+t} = \frac{1}{\omega - x - t},\quad E(T_x) = \mathring{e}_x = \frac{\omega - x}{2}.$$

Often need to recognize this distribution from ${}_tp_x$.

12] The curtate duration at failure RV

$K_x = \lfloor T_x \rfloor$ where $\lfloor 7.8 \rfloor = 7$.

$K_x = k$ means person with T_x died in the time interval $(k, k+1)$.



at time $t=40$, if $x=40$

T_0 is 40 but $T_{40} = 0$

So person with T_0 died in time interval $(x+k, x+k+1)$, given $T_0 > x$.

13] The pmf of K_x is $P(K_x = k) = p_{K_x}(k) =$

$${}_{k|}q_x = P(k < T_x < k+1) = P(x+k < T_0 < x+k+1 \mid T_0 > x)$$

$$= {}_k p_x - {}_{k+1}p_x = F_x(k+1) - F_x(k) = S_x(k) - S_x(k+1).$$

The curtate expectation of life at age x is $e_x = E[K_x]$.

$E(T_x) = \mathring{e}_x \approx e_x + 0.5$.

let $W = X | Y =$

So $T_x =$ $T_0 | T_0 >$

14) The probability that (x) will die between $x+n$ and $x+n+m$ is

$$_{n|m}q_x = P(x+n < T_0 \le x+n+m \mid T_0 > x)$$

$$= {}_np_x - {}_{n+m}p_x = {}_{n+m}q_x - {}_nq_x = {}_np_x \; {}_mq_{x+n}.$$

For $m=1$, $_{n|1}q_x = {}_{n|}q_x = P(K_x = n)$

$$= P(x+n < T_0 < x+n+1 \mid T_0 > x).$$

15) multiplication rule: $_{n+m}p_x = {}_np_x \; {}_mp_{x+n}$

"angle n"

16) $\mathring{e}_{x:\overline{n}|}$ = expected number of years lived in $(x, x+n]$ by a randomly selected survivor to age x, and

$$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \, dt = \int_0^n S_x(t) \, dt$$

ch 4:

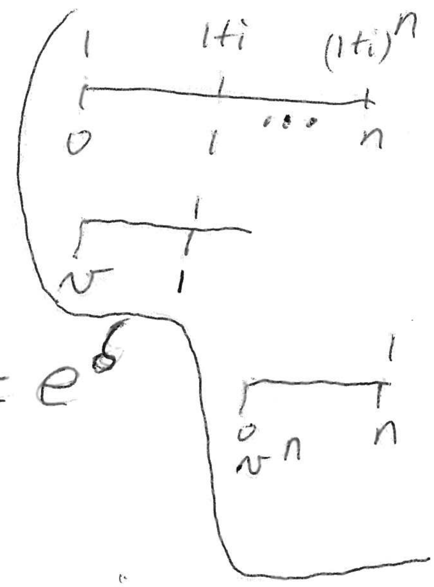
3.5

17) i) The compound interest factor

$v = \frac{1}{1+i}$ and $0 < v < 1$.

ii) The effective rate of interest

$i = \frac{1-v}{v} > 0$.



iii) The force of interest

$\delta = \log(1+i) > 0$, $1+i = e^{\delta}$

Know: so $v = e^{-\delta}$.

iv) The effective rate of discount $d = \frac{i}{1+i} = iv$
$= 1-v > 0$.

18) A life insurance model has benefit function b_t and a discount function v_t where t = length of time from issue of insurance until death.

Often $v_t = v^t$ and $b_t = 1$ unit.

The present value function $z_t = b_t v_t$.

The present value RV $Z = z_{T_x}$
or $Z = z_{K_x+1}$. Often let $T = T_x$.

19] $E(Z)$ = actuarial present value (APV) = expected present value (EPV) = net single premium (NSP).

Z, A for insurance

20] Formulas are usually given for unit payment. Let $A = E(Z)$ and ${}^2A = E(Z^2)$. For nonunit payment c, multiply the unit payment formula for A by c and the unit payment formula for ${}^2A$ by c^2.

21] Discrete insurance

i) whole life $\quad z_t = v^t, \; t \geq 0$

$$A_x = E(Z_x) = E(v^{K_x+1})$$

$${}^2A_x = E(Z_x^2)$$

ii) n year term $\quad z_t = v^t \, \overbrace{I(t \leq n)}^{\begin{cases} 1 & t \leq n \\ 0 & \text{else} \end{cases}}$

$$Z^{\,1}_{x:\overline{n}|} = v^{K_x+1} I(K_x < n)$$

4.5

decorations

$$A^{1}_{x:\overline{n}|} = E(Z^{1}_{x:\overline{n}|})$$

$$^{2}A^{1}_{x:\overline{n}|} = E[(Z^{1}_{x:\overline{n}|})^2]$$

iii) n year deferred

$$Z_t = v^t \, I(t > n)$$

$$_{n|}Z_x = v^{K_x+1} I(K_x \geq n)$$

$$_{n|}A_x = E(_{n|}Z_x)$$

$$^{2}{}_{n|}A_x = E[(_{n|}Z_x)^2]$$

iv) n year endowment

$$Z_t = \begin{cases} v^t & t \leq n \\ v^n & t > n \end{cases} = v^{\min(t,n)}$$

$$Z_{x:\overline{n}|} = \begin{cases} v^{K_x+1}, & K_x < n \\ v^n, & K_x \geq n \end{cases}$$

$$A_{x:\overline{n}|} = E(Z_{x:\overline{n}|}), \quad ^{2}A_{x:\overline{n}|} = E[(Z_{x:\overline{n}|})^2]$$

v) discrete = continuous pure endowment insurance

$$Z_t = v^n \, I(t > n)$$

$$Z_{x:\overset{1}{\overline{n|}}} = \underbrace{v^n}_{e^{-\delta n}} I(T_x > n)$$

$$* \; A_{x:\overset{1}{\overline{n|}}} = E\left(Z_{x:\overset{1}{\overline{n|}}}\right) = e^{-\delta n} S_x(n) = e^{-\delta n} {}_nP_x$$

$= {}_nE_x$ easy to compute

$${}^2A_{x:\overset{1}{\overline{n|}}} = E\left[\left(Z_{x:\overset{1}{\overline{n|}}}\right)^2\right] = e^{-2\delta n} S_x(n) = e^{-2\delta n} {}_nP_x$$

$$V\left(Z_{x:\overset{1}{\overline{n|}}}\right) = e^{-2\delta n} {}_nP_x \, {}_nq_x = e^{-2\delta n} S_x(n) \, F_x(n)$$

22] $\underset{\text{whole}}{Z_x} = \underset{\text{term}}{Z^1_{x:\overline{n|}}} + \underset{\text{deferred}}{{}_{n|}Z_x}$, $A_x = A^1_{x:\overline{n|}} + {}_{n|}A_x$

$$(Z_x)^2 = \left(Z^1_{x:\overline{n|}}\right)^2 + \left({}_{n|}Z_x\right)^2, \quad {}^2A_x = {}^2A^1_{x:\overline{n|}} + {}^2{}_{n|}A_x$$

$\underset{\text{n year endowment}}{Z_{x:\overline{n|}}} = \underset{\text{term}}{Z^1_{x:\overline{n|}}} + \underset{\text{pure endowment}}{Z_{x:\overset{1}{\overline{n|}}}}$, $A_{x:\overline{n|}} = A^1_{x:\overline{n|}} + A_{x:\overset{1}{\overline{n|}}}$

$$\left(Z_{x:\overline{n|}}\right)^2 = \left(Z^1_{x:\overline{n|}}\right)^2 + \left(Z_{x:\overset{1}{\overline{n|}}}\right)^2, \quad {}^2A_{x:\overline{n|}} = {}^2A^1_{x:\overline{n|}} + {}^2A_{x:\overset{1}{\overline{n|}}}$$

23] continuous insurance

note that 21 v) n year pure endowment insurance is both discrete and continuous insurance.

5.5

i) whole life $\quad Z_t = v^t, \; t > 0$

$T = T_x$

$$\bar{Z}_x = v^T$$

$$\bar{A}_x = E(\bar{Z}_x) = E(e^{-\delta T}) = \int_0^\infty e^{-\delta t} f_x(t)\,dt = g(\delta)$$

$${}^2\bar{A}_x = E(\bar{Z}_x^2) = E(e^{-2\delta T}) = \int_0^\infty e^{-2\delta t} f_x(t)\,dt = g(2\delta)$$

ii) n year term $\quad Z_t = v^t \, I(t \le n)$

$$\bar{Z}^{\,1}_{x:\overline{n}|} = v^T \, I(T \le n)$$

bar for continuous

$$\bar{A}^{\,1}_{x:\overline{n}|} = E(\bar{Z}^{\,1}_{x:\overline{n}|}) = \int_0^n e^{-\delta t} f_x(t)\,dt$$

$${}^2\bar{A}^{\,1}_{x:\overline{n}|} = E\left[(\bar{Z}^{\,1}_{x:\overline{n}|})^2\right] = \int_0^n e^{-2\delta t} f_x(t)\,dt$$

iii) n year deferred $\quad Z_t = v^t \, I(t > n)$

$${}_{n|}\bar{Z}_x = v^T \, I(T > n)$$

$${}_{n|}\bar{A}_x = E({}_{n|}\bar{Z}_x) = \int_n^\infty e^{-\delta t} f_x(t)\,dt$$

$${}^2{}_{n|}\bar{A}_x = E\left[({}_{n|}\bar{Z}_x)^2\right] = \int_n^\infty e^{-2\delta t} f_x(t)\,dt$$

(v) n year endowment $z_t = \begin{cases} v^t & t \le n \\ v^n & t > n \end{cases} = v^{\min(t,n)}$ M402 6

$$\bar{Z}_{x:\overline{n}|} = \begin{cases} v^T & T \le n \\ v^n & T > n \end{cases}$$

← pure endowment does not get a bar since it is discrete and continuous

$$\bar{Z}_{x:\overline{n}|} = \bar{Z}^{\,1}_{x:\overline{n}|} + Z_{x:\overset{1}{\overline{n}|}}$$

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{\,1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$$

$$\left(\bar{Z}_{x:\overline{n}|}\right)^2 = \left(\bar{Z}^{\,1}_{x:\overline{n}|}\right)^2 + \left(Z_{x:\overset{1}{\overline{n}|}}\right)^2$$

$$^2\bar{A}_{x:\overline{n}|} = {}^2\bar{A}^{\,1}_{x:\overline{n}|} + {}^2A_{x:\overset{1}{\overline{n}|}}$$

24]* If $T = T_x \sim \text{Exponential}(\mu)$,

then $\int_0^\infty \mu e^{-\mu t}\,dt = 1$ so $\int_0^\infty e^{-\mu t}\,dt = \frac{1}{\mu}$, $\mu > 0$.

If $b_t = e^{\theta t}$ and $v_t = v^t$, then $z_t = e^{\theta t} e^{-\delta t}$

and $\bar{Z} = z_T b_T$ and $\bar{A} = E(\bar{Z}) = E(e^{\theta T} e^{-\delta T})$

$$\stackrel{E}{=} \int_0^\infty e^{\theta t} e^{-\delta t} \mu e^{-\mu t}\,dt = \mu \int_0^\infty e^{-t(\mu+\delta-\theta)}\,dt$$

$$\stackrel{E}{=} \frac{\mu}{\mu+\delta-\theta} \quad \text{if } \mu+\delta-\theta > 0 \text{, and}$$

$$E\left[(\bar{Z})^j\right] \overset{E}{=} \int_0^\infty \left(e^{\theta t} e^{-\delta t}\right)^j \mu e^{-\mu t}\, dt =$$

$$\mu \int_0^\infty e^{-t(\mu+\delta j-\theta j)}\, dt \overset{E}{=} \frac{\mu}{\mu+\delta j-\theta j} \quad \text{if}$$

$\mu+\delta j-\theta j > 0$. Note $\theta = 0$ corresponds

* to unit payment: $\bar{A}_x \overset{E}{=} \frac{\mu}{\mu+\delta}$, ${}^2\bar{A}_x \overset{E}{=} \frac{\mu}{\mu+2\delta}$

25] often use $\int_0^n$ or $\int_n^\infty$ instead of $\int_0^\infty$

ch5 26] $a_{\overline{n}|} = \sum_{j=1}^{n} v^j = \frac{1-v^n}{i}$

$$\ddot{a}_{\overline{n}|} = \sum_{j=0}^{n-1} v^j = \frac{1-v^n}{d}$$

$$\bar{a}_{\overline{n}|} = \int_0^n v^t\, dt = \frac{1-v^n}{\delta}.$$

a for annuity, Y

27] i) A discrete immediate whole life annuity pays 1 unit at times 1, 2, ... as long as (x) survives. The present value RV

$$Y_x = a_{\overline{K_x}|} = \frac{1}{i}\left(1-(1+i)Z_x\right).$$

APV $= a_x = E\,Y_x$, $\quad V(Y_x) = \frac{{}^2A_x - (A_x)^2}{d^2}$