(a) from 125)

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b) $_{n|m}q_{xy} = {}_np_{xy}\ {}_mq_{x+n:y+n} = {}_np_{xy} - {}_{n+m}p_{xy}$

c) $_{n+m}p_{xy} = {}_np_{xy}\ {}_mp_{x+n:y+n}$

Think of $xy = w$. Then many formulas apply, but $w+n \to x+n:y+n$.

w "survives" as long as both x and y survive. Status w "ends" as soon as one of x or y dies.

joint survival:

$_tp_{xy}$ = P[(x) and (y) are both alive in t years]

$_tq_{xy}$ = P[(x) and (y) are not both alive in t years]

last survival

$_tp_{\overline{xy}}$ = P[at least one of (x) and (y) is alive in t years]

$_tq_{\overline{xy}}$ = P[(x) and (y) are both dead in t years]

q type probabilities are associated with failure of status.

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The joint life status xy fails on the first death of (x) and (y).

The last survival status fails on the last death of (x) and (y).

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18] * p297, 00 Even if T_x and T_y are dependent,

$$s_{\overline{xy}}(t) = {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy}$$

19] pdf $f_{\overline{xy}}(t) = -\frac{d}{dt} s_{\overline{xy}}(t) = -\frac{d}{dt}\left({}_tp_x + {}_tp_y - {}_tp_{xy}\right)$

$$= -\frac{d}{dt} s_x(t) - \frac{d}{dt} s_y(t) + \frac{d}{dt} s_{xy}(t) =$$

$$f_x(t) + f_y(t) - f_{xy}(t) =$$

$$({}_tp_x)(\mu_{x+t}) + ({}_tp_y)(\mu_{y+t}) - ({}_tp_{xy})\underbrace{(\mu_{x+t:y+t})}_{\mu_{x+t}\,\mu_{y+t} \text{ if } T_x \perp\!\!\!\perp T_y}$$

Since the $z = \overline{xy}$ status ends only after both x and y die, many single status formulas for z do not hold.

20] $\mu_{\overline{xy}}(t) = \dfrac{f_{\overline{xy}}(t)}{s_{\overline{xy}}(t)} = \dfrac{({}_tp_x)(\mu_{x+t}) + ({}_tp_y)(\mu_{y+t}) - ({}_tp_{xy})(\mu_{x+t:y+t})}{{}_tp_x + {}_tp_y - {}_tp_{xy}}$

If $T_x \perp\!\!\!\perp T_y$, $\mu_{\overline{xy}}(t) = \dfrac{{}_tq_x\,{}_tp_y\,\mu_{y+t} + {}_tq_y\,{}_tp_x\,\mu_{x+t}}{{}_tp_{\overline{xy}}}$.

21] Let $K_{\overline{xy}}$ be the curtate duration at failure RV for status $(\overline{xy})$

a) ${}_{n|}q_{\overline{xy}} = P(n < T_{\overline{xy}} \le n+1) = P(K_{\overline{xy}} = n)$

$$= {}_np_{\overline{xy}} - {}_{n+1}p_{\overline{xy}} = ({}_np_x + {}_np_y - {}_np_{xy}) - ({}_{n+1}p_x + {}_{n+1}p_y - {}_{n+1}p_{xy})$$

$= {}_{n|}q_x + {}_{n|}q_y - {}_{n|}q_{xy}$. b) ${}_{n|m}q_{\overline{xy}} = {}_np_{\overline{xy}} - {}_{n+m}p_{\overline{xy}}$.

22] p298 * $\overset{\circ}{e}_{\overline{xy}} = E[T_{\overline{xy}}] = \int_0^\infty t\, f_{T_{\overline{xy}}}(t)\,dt = \int_0^\infty {}_tp_{\overline{xy}}\,dt$

$= \overset{\circ}{e}_x + \overset{\circ}{e}_y - \overset{\circ}{e}_{xy}$. RHS is often easier to compute ~~than the~~ integrals.

23] p299 $e_{\overline{xy}} = E[K_{\overline{xy}}] = \sum_{k=1}^{\infty} {}_kp_{\overline{xy}} = e_x + e_y - e_{xy}$.

24] The temporary curtate expectation

$$e_{\overline{xy}:\overline{n|}} = \sum_{k=1}^{n} {}_k p_{\overline{xy}} = e_{x:\overline{n|}} + e_{y:\overline{n|}} - e_{xy:\overline{n|}}$$

= average # of whole years of survival within the next n years of the last survivor status $(\overline{xy})$.

25] $T_{xy} = \min(T_x, T_y)$ and $T_{\overline{xy}} = \max(T_x, T_y)$.

So T_{xy} is one of T_x, T_y and $T_{\overline{xy}}$ is the other.
Hence if $g(a,b) = g(b,a)$, then $g(T_{xy}, T_{\overline{xy}}) = g(T_x, T_y)$.
Thus $T_{xy} + T_{\overline{xy}} = T_x + T_y$,

and $(T_{xy})(T_{\overline{xy}}) = (T_x)(T_y)$.

So $T_{\overline{xy}} = T_x + T_y - T_{xy}$ and $\mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}$.

Similarly $P(T_{xy} > t) + P(T_{\overline{xy}} > t) = P(T_x > t) + P(T_y > t)$,

so ${}_t p_{xy} + {}_t p_{\overline{xy}} = {}_t p_x + {}_t p_y$.

So ${}_t p_{\overline{xy}} = {}_t p_x + {}_t p_y - {}_t p_{xy}$.

(so ${}_t q_{\overline{xy}} = {}_t q_x + {}_t q_y - {}_t q_{xy}$ change > to ≤)

26] $\mathrm{cov}(T_{xy}, T_{\overline{xy}}) > 0$.

27] p296 If $T_x \sim EXP(\mu_x) \perp\!\!\!\perp T_y \sim EXP(\mu_y)$ then

Know $T_{xy} = \min(T_x, T_y) \sim EXP(\mu_x + \mu_y)$. So $\mathring{e}_{xy} = \frac{1}{\mu_x + \mu_y}$.

Proof $S_{T_{xy}}(t) = {}_t p_{xy} = {}_t p_x \, {}_t p_y = e^{-t\mu_x} e^{-t\mu_y} = \underbrace{e^{-t(\mu_x + \mu_y)}}_{\text{Survival fn for } EXP(\mu_x+\mu_y) \text{ RV}}$

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28] Know notation: If x and y are numbers, (xy) is denoted $(40:50)$ and $(x+n \ \ y+n)$ as $(x+n : y+n)$.

$(\overline{xy})$ as $(\overline{40:50})$ and $(\overline{x+n \ \ y+n})$ as $(\overline{x+n : y+n})$.

So subscripts usually have a colon and sometimes bars. Think of (xy) as $(x:y)$ and $(\overline{xy})$ as $(\overline{x:y})$.

See ex 12.2 and 12.5

29] $\min(x, j) = \begin{cases} x & x \le j \\ j & x > j \end{cases}$

$$\text{So } E[\min(X, j)] = \int_0^j x f_X(x)\,dx + \int_j^\infty j f_X(x)\,dx$$

$$= \int_0^j x f_X(x)\,dx + j \underbrace{P(X > j)}_{S_X(j)}.$$

30] $\max(x, j) = \begin{cases} j & x \le j \\ x & x > j \end{cases}$

$$\text{So } E[\max(X, j)] = \int_0^j j f_X(x)\,dx + \int_j^\infty x f_X(x)\,dx$$

$$= j F_X(j) + \int_j^\infty x f_X(x)\,dx.$$

31] $_{n|}q_{\overline{xy}} = P(n < T_{\overline{xy}} \le n+1) = {}_np_{\overline{xy}} - {}_{n+1}p_{\overline{xy}}$

$= {}_{n|}q_x + {}_{n|}q_y - {}_{n|}q_{xy} = P(K_{\overline{xy}} = n)$

$P(n < T_{\overline{xy}} < n+1)$
$+ P(n < T_{xy} < n+1)$
$= P(n < T_x < n+1)$
$+ P(n < T_y < n+1)$

§12.3 32] $P[(x) \text{ fails before } (y)] = P(T_x < T_y)$. 12.5

If $T_x \perp\!\!\!\perp T_y$, $P(T_x < T_y) = \int_0^\infty S_y(t)\, f_x(t)\, dt$

$= {}_\infty q^1_{xy} = \int_0^\infty {}_tp_y \underbrace{({}_tp_x)(\mu_{x+t})}_{f_{T_x}(t)}\, dt \qquad = \int_0^\infty {}_tp_{xy}\, \mu_{x+t}\, dt$

↑ the 1 means (x) fails before (y)

33] ${}_\infty q^1_{xy} = E[S_y(T_x)]$ if $T_x \perp\!\!\!\perp T_y$.

34] ${}_\infty q^1_{xy} = \int_0^\infty \int_t^\infty f_{T_x,T_y}(t,s)\, ds\, dt = \int_0^\infty \int_t^\infty f_{T_y|T_x}(s|t)\, ds\, f_{T_x}(t)\, dt$

$\left(f_{T_x,T_y}(t,s) = \underbrace{f_{T_y|T_x}(s|t)}\, f_{T_x}(t)\right)$

$\left(\text{conditional} = \frac{\text{joint}}{\text{marginal}}\right)$

$= \int_0^\infty P(T_y > t \mid T_x = t)\, f_{T_x}(t)\, dt$.

If $T_x \perp\!\!\!\perp T_y$, then ${}_\infty q^1_{xy} = \int_0^\infty P(T_y > t)\, f_{T_x}(t)\, dt = \int_0 S_y(t)\, f_x(t)\, dt$.

35] If $T_x \perp\!\!\!\perp T_y$, $P[(x) \text{ fails before } (y) \text{ and within } n \text{ years}]$

$= {}_nq^1_{xy} = \int_0^n {}_tp_{xy}\, \mu_{x+t}\, dt = \int_0^n S_y(t)\, f_x(t)\, dt$

(1st failure is (x))

36] If $T_x \perp\!\!\!\perp T_y$, $P[(y) \text{ fails before } (x) \text{ and within } n \text{ years}]$

$= {}_nq_{x\overset{1}{y}} = \int_0^n {}_tp_{xy}\, \mu_{y+t}\, dt = \int_0^n S_x(t)\, f_y(t)\, dt$

37] If $T_x \perp\!\!\!\perp T_y$, $P(T_x > T_y) = {}_\infty q^2_{xy} = 1 - {}_\infty q^1_{xy} = P[(x) \text{ fails after } (y)]$

$= \int_0^\infty F_{T_y}(t)\, f_{T_x}(t)\, dt = E[F_{T_y}(T_x)]$

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38] If $T_x \perp\!\!\!\perp T_y$, P((x) fails after (y) and within n years)

$= {}_nq^{\;2}_{x\,y} = \int_0^n F_{T_y}(t)\, f_{T_x}(t)\,dt = {}_nq_x - {}_nq^{\;1}_{x\,y}$
(2nd failure is (x))

39] If $T_x \perp\!\!\!\perp T_y$, P((y) fails after (x) and within n years)

$= {}_nq_{x\,y}^{\;\;\;2} = \int_0^n F_{T_x}(t)\, f_{T_y}(t)\,dt = {}_nq_y - {}_nq_{x\,y}^{\;\;\;1}$

40] ${}_nq^{\;1}_{x\,y} + {}_nq_{x\,y}^{\;\;\;1} = {}_nq_{xy}$

P(1st failure is (x) and before n) + P(1st failure is (y) and before n)
= P(1st failure is before n)

41] ${}_nq^{\;2}_{x\,y} + {}_nq_{x\,y}^{\;\;\;2} = {}_nq_{\overline{xy}}$

P(last failure is (x) and before n) + P(last failure is (y) and before n)
= P(last failure is before n)

42] * $E(T_{xy}^2) = 2\int_0^\infty t\;{}_tp_{xy}\,dt$, $E(T_{\overline{xy}}^2) = E[(T_{\overline{xy}})^2] = 2\int_0^\infty t\;{}_tp_{\overline{xy}}\,dt$.

§12.4 43] Insurance, annuities and Pensions for (xy) and ($\overline{xy}$) are like those for (x) but replace T_x, K_x, A_x, Z_x etc by T_{xy} or $T_{\overline{xy}}$, ..., Z_{xy} or $Z_{\overline{xy}}$ etc.

44] p303 discrete whole life insurance for (xy), $T_x \perp\!\!\!\perp T_y$

$Z_{xy} = v^{1+K_{xy}}$, $A_{xy} = E[Z_{xy}] = \sum_{k=0}^{\infty} v^{k+1} \underbrace{({}_{k|}q_{xy})}_{P(K_{xy}=k)}$

${}^2A_{xy} = E[(Z_{xy})^2] = \sum_{k=0}^{\infty} v^{2(k+1)} ({}_{k|}q_{xy})$

45) other discrete insurance models from ch4 are similar.

46] continuous whole life insurance for (xy) $T_x \perp\!\!\!\perp T_y$

$$\bar{Z}_{xy} = v^{T_{xy}}, \quad \bar{A}_{xy} = E[\bar{Z}_{xy}] = \int_0^\infty e^{-\delta t} f_{xy}(t)\,dt$$

$T_x\ T_y$

$$= \int_0^\infty e^{-\delta t}\ {}_tp_{xy}\ \mu_{x+t:y+t}\,dt$$

$$^2\bar{A}_{xy} = E[(\bar{Z}_{xy})^2] = \int_0^\infty e^{-2\delta t} f_{xy}(t)\,dt = \int_0^\infty e^{-2\delta t}\ {}_tp_{xy}\ \mu_{x+t:y+t}\,dt$$

47) $T_x \perp\!\!\!\perp T_y$, discrete whole life insurance for $(\overline{xy})$

$$A_{\overline{xy}} = E(Z_{\overline{xy}}) = \sum_{k=0}^\infty v^{k+1} ({}_{k|}q_{\overline{xy}})$$

$$^2A_{\overline{xy}} = E(Z^2_{\overline{xy}}) = \sum_{k=0}^\infty v^{2(k+1)} \underbrace{({}_{k|}q_{\overline{xy}})}_{P(K_{\overline{xy}}=k)}$$

$*\quad A_{\overline{xy}} = A_x + A_y - A_{xy}$

48) $T_x \perp\!\!\!\perp T_y$, continuous whole life insurance for $(\overline{xy})$

$$\bar{A}_{\overline{xy}} = E(\bar{Z}_{\overline{xy}}) = \int_0^\infty e^{-\delta t} f_{\overline{xy}}(t)\,dt$$

$$* \quad = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

$$^2\bar{A}_{\overline{xy}} = E[(\bar{Z}_{\overline{xy}})^2] = \int_0^\infty e^{-2\delta t} f_{\overline{xy}}(t)\,dt$$

49] $T_x \perp\!\!\!\perp T_y$, Annuity models sub (xy) or $(\overline{xy})$ for (x).

i) discrete annual immediate whole life annuity for (xy)

$$a_{xy} = E(Y_{xy}) = \sum_{k=1}^\infty v^k ({}_kp_{xy})$$

ii) discrete annual whole life annuity-due for $(\overline{xy})$

$$\ddot{a}_{\overline{xy}} = \sum_{k=0}^\infty v^k ({}_kp_{\overline{xy}})$$

iii) continuous temporary n year annuity for (xy)

$$\bar{a}_{xy:\overline{n}|} = E(\bar{Y}_{xy:\overline{n}|}) = \int_0^n e^{-\delta t}\ {}_tp_{xy}\,dt = \int_0^n e^{-\delta t} S_{xy}(t)\,dt$$

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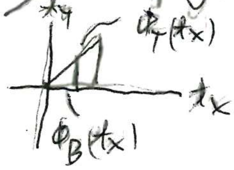
50] Other relationships also hold

$$A_{xy} = 1 - d\, \ddot{a}_{xy}, \quad \bar{A}_{\overline{xy}} = 1 - \delta\, \bar{a}_{\overline{xy}} \quad \text{etc}$$

Skip § 12.4.3 for now.

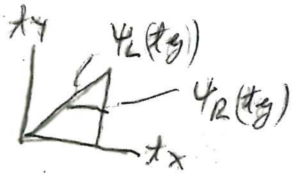
§ 12.6 51) Let $f_{T_x, T_y}(t_x, t_y)$ be the joint pdf of T_x and T_y.

52) marginals



$$f_{T_x}(t_x) = \int_{-\infty}^{\infty} f_{T_x, T_y}(t_x, t_y)\, dt_y = \int_{\phi_B(t_x)}^{\phi_T(t_x)} f_{T_x, T_y}(t_x, t_y)\, dt_y$$

$$f_{T_y}(t_y) = \int_{-\infty}^{\infty} f_{T_x, T_y}(t_x, t_y)\, dt_x = \int_{\psi_L(t_y)}^{\psi_R(t_y)} f_{T_x, T_y}(t_x, t_y)\, dt_x$$

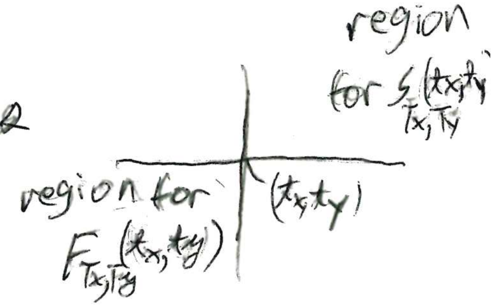


See Math 483

53] $E[T_x T_y] = \int_0^\infty \int_0^\infty t_x t_y\, f_{T_x, T_y}(t_x, t_y)\, dt_x\, dt_y$

54] $F_{T_x, T_y}(t_x, t_y) = \int_0^{t_y} \int_0^{t_x} f_{T_x, T_y}(r, s)\, dr\, ds.$

55] $S_{T_x, T_y}(t_x, t_y) = \int_{t_y}^{\infty} \int_{t_x}^{\infty} f_{T_x, T_y}(r, s)\, dr\, ds$



note that $F_{T_x, T_y}(t_x, t_y) \neq 1 - S_{T_x, T_y}(t_x, t_y)$

56] $S_{T_x, T_y}(n, n) = P(T_{xy} > n) = {}_n p_{xy}$ (joint life status)

57] $F_{T_x, T_y}(n, n) = P(T_{\overline{xy}} \leq n) = {}_n q_{\overline{xy}}$ (last survivor status)

58)* Let $T_{x_1}, T_{x_2}, \dots, T_{x_m}$ be ind $EXP(\mu_i)$ RVs. 14.5

Let $T = T_{x_1 \dots x_m} = \min(T_{x_1}, \dots, T_{x_m}) \sim EXP\left(\sum_{i=1}^m \mu_i\right)$

and $u = (x_1 \dots x_m)$ be the joint life status.

$m=1,\ u=(x)$

$m=2,\ u=(x_1 x_2) = (xy)$

i) $S_T(t) = P(T>t) = e^{-t\sum_{i=1}^m \mu_i}$

ii) $F_T(t) = P(T \le t) = 1 - e^{-t\sum_{i=1}^m \mu_i}$

iii) $f_T(t) = \left(\sum_{i=1}^m \mu_i\right) e^{-t\sum_{i=1}^m \mu_i}$

iv) $\mu_T(t) = \sum_{i=1}^m \mu_i$

v) $\overset{\circ}{e}_u = E(T) = \dfrac{1}{\sum_{i=1}^m \mu_i}$

vi) whole life insurance

$\bar{Z}_u = v^T,\quad \bar{A}_u = \int_0^\infty e^{-\delta t}\left(\sum_{i=1}^m \mu_i\right) e^{-t\sum_{i=1}^m \mu_i}\,dt = \dfrac{\sum_{i=1}^m \mu_i}{\delta + \sum_{i=1}^m \mu_i} = E(\bar{Z}_u)$

$^2\bar{A}_u = \int_0^\infty e^{-2\delta t}\left(\sum_{i=1}^m \mu_i\right) e^{-t\sum_{i=1}^m \mu_i}\,dt = \dfrac{\sum_{i=1}^m \mu_i}{2\delta + \sum_{i=1}^m \mu_i} = E\left[(\bar{Z}_u)^2\right]$

If $\alpha = P(\bar{Z}_u \le \xi_\alpha)$ then solve $\alpha = \exp\left(\dfrac{\log(\xi_\alpha)}{\delta} \sum_{i=1}^m \mu_i\right)$ for

$\xi_\alpha = \exp\left[\dfrac{\delta}{\sum_{i=1}^m \mu_i} \log(\alpha)\right]$

vii) If $b_t = e^{\theta t}$, $E\left[(\bar{Z})^j\right] = \int_0^\infty e^{\theta j t} e^{-\delta j t}\left(\sum_{i=1}^m \mu_i\right) e^{-t\sum_{i=1}^m \mu_i}\,dt$

$= \dfrac{\sum_{i=1}^m \mu_i}{\sum_{i=1}^m \mu_i + \delta j - \theta j}$ if $\sum_{i=1}^m \mu_i + \delta j - \theta j > 0$

viii) whole life annuity $\bar{a}_u = E(\bar{Y}_u) = \int_0^\infty e^{-\delta t} e^{-t\sum_{i=1}^m \mu_i}\,dt = \dfrac{1}{\delta + \sum_{i=1}^m \mu_i}$

$V(\bar{Y}_u) = \dfrac{^2\bar{A}_u - (\bar{A}_u)^2}{\delta^2}$

ix) $\longrightarrow V(T) = \left(\dfrac{1}{\sum_{i=1}^m \mu_i}\right)^2$

59] p294–300 Since $T_{xy} + T_{\overline{xy}}$ $(\min(T_x, T_y) + \max(T_x, T_y))$ $= T_x + T_y$ 402 15
49 59

and $T_{xy}\, T_{\overline{xy}} = T_x\, T_y$, repeating earlier work

$T_{\overline{xy}} = T_x + T_y - T_{xy}$. Taking expectations

gives $\mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}$.

Similarly ${}_tp_{xy} + {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y$

$(P[\min(T_x, T_y) > t] + P[\max(T_x, T_y) > t] = P(T_x > t) + P(T_y > t))$

one of these is T_x and one is T_y

60] p300 $\text{Cov}(T_{xy}, T_{\overline{xy}}) = E(T_{xy}\, T_{\overline{xy}}) - E(T_{xy})E(T_{\overline{xy}})$

$= E(T_x T_y) - E(T_{xy})\, E(T_x + T_y - T_{xy})$.

If $T_x \perp\!\!\!\perp T_y$, then $E(T_x T_y) = E(T_x)\, E(T_y)$.

So $\text{cov}(T_{xy}, T_{\overline{xy}}) = \mathring{e}_x \mathring{e}_y - \mathring{e}_{xy}(\mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy})$

$= \dots = (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy}) > 0$.

61] A generalized De Moivre $GD(\alpha, \theta)$ distribution has survival

15.5

function $S_0(t) = \left(\frac{\theta - t}{\theta}\right)^{\alpha}$ for $0 < t < \theta$,
where $\alpha > 0$.
often $\theta = \omega - x$.

If $T_x \sim$ DeMoivre$(\omega - x)$, then $\alpha = 1$.
If $T_x \sim GD(\omega - x, \alpha)$, then for $0 < t < \omega - x$.

See HW3 1

$$S_x(t) = {}_tP_x = \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha}$$

$$F_x(t) = {}_tq_x = 1 - \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha}$$

$$f_x(t) = {}_tP_x \, \mu_{x+t} = \frac{\alpha(\omega - x - t)^{\alpha - 1}}{(\omega - x)^{\alpha}}$$

$$\mu_x(t) = \mu_{x+t} = \frac{\alpha}{\omega - x - t}$$

$$E(T_x) = \mathring{e}_x = \frac{\omega - x}{\alpha + 1}$$

$$V(T_x) = \frac{\alpha(\omega - x)^2}{(\alpha + 1)^2(\alpha + 2)}$$