see HW3 #1 and E[illegible]rev #63 401 60 $T_x \sim GD(\alpha, \omega - x)$

$$E(T_x^2) = 2\int_0^{\omega-x} t \;\, {}_tp_x \, dt = \frac{2}{(\omega-x)^\alpha}\int_0^{\omega-x} t \;\, (\omega-x-t)^\alpha \, dt$$

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$$\left\{\begin{array}{l} u = \omega - x - t, \quad du = -dt, \quad t = 0 \to u = \omega - x, \quad t = \omega - x \to u = 0 \\ t = \omega - x - u \end{array}\right.$$

$$= -\frac{2}{(\omega-x)^\alpha}\int_{\omega-x}^{0} (\omega - x - u) \; u^\alpha \, du$$

$$= \frac{2}{(\omega-x)^\alpha}\left[\int_0^{\omega-x} (\omega-x) u^\alpha \, du - \int_0^{\omega-x} u^{\alpha+1} \, du\right]$$

$$= \frac{2}{(\omega-x)^{\alpha-1}} \left.\frac{u^{\alpha+1}}{\alpha+1}\right|_0^{\omega-x} - \frac{2}{(\omega-x)^\alpha} \left.\frac{u^{\alpha+2}}{\alpha+2}\right|_0^{\omega-x}$$

$$= \frac{2}{\alpha+1}(\omega-x)^2 - \frac{2}{\alpha+2}(\omega-x)^2 = (\omega-x)^2 \, 2\left(\frac{1}{\alpha+1} - \frac{1}{\alpha+2}\right)$$

$$= \frac{2(\omega-x)^2}{(\alpha+1)(\alpha+2)}. \quad V(T_x) = E(T_x^2) - (E\,T_x)^2$$

$$= \frac{2(\omega-x)^2}{(\alpha+1)(\alpha+2)} - \left(\frac{\omega-x}{\alpha+1}\right)^2 = \frac{2(\omega-x)^2(\alpha+1) - (\omega-x)^2(\alpha+2)}{(\alpha+1)^2(\alpha+2)}$$

$$= \frac{(\omega-x)^2\alpha}{(\alpha+1)(\alpha+2)}$$

✓ since $(\omega-x)^2\left[2\alpha+2-\alpha-2\right]$

$= (\omega-x)^2\alpha.$

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62] If T_{x_i} are ind $GD(\alpha_i, \omega - x)$,

then $T_{x_1 x_2 \ldots x_K} \sim GD\left(\sum_{i=1}^{K} \alpha_i, \omega - x\right)$.

So if $T_x \sim GD(\alpha_x, \omega - x) \perp\!\!\!\perp T_y \sim GD(\alpha_y, \omega - x)$,

then $T_{xy} \sim GD(\alpha_x + \alpha_y, \omega - x)$ since

$$ {}_tp_{xy} = {}_tp_x \, {}_tp_y = \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha_x} \left(\frac{\omega - \cancel{x} - t}{\omega - x}\right)^{\alpha_y} = \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha_x + \alpha_y} $$

for $0 < t < \omega - x$.

So if $T_x \sim U(0, \omega - x) \perp\!\!\!\perp T_y \sim U(0, \omega - x)$,

DeMoivre $(\omega - x) \sim GD(1, \omega - x)$

$T_{xy} \sim GD(2, \omega - x)$ since $\alpha_x = \alpha_y = 1$.

Note: need $\omega_x - x = \omega_y - y \equiv \omega - x$.

63] If T_{x_i} are ind $EXP(\mu_{x_i})$, then

$T_{x_1 x_2 \ldots x_K} \sim EXP\left(\sum_{i=1}^{K} \mu_{x_i}\right)$ since

$$ {}_tp_{x_1 x_2 \ldots x_K} = \prod_{i=1}^{K} {}_tp_{x_i} = \prod_{i=1}^{K} e^{-t\mu_{x_i}} = e^{-t \sum_{i=1}^{K} \mu_{x_i}} $$

for $t > 0$.

17.5

ex] $\overset{T_x}{T_{30}} \perp\!\!\!\perp \overset{T_Y}{T_{60}}$

$${}_tp_{30} = \frac{70-t}{70} \qquad 0<t<70$$

$${}_tp_{60} = \frac{50-t}{50} \qquad 0<t<50$$

(Demoivre but $\omega_x - x \neq \omega_y - y$)

so $${}_tp_{30:60} = {}_tp_{30}\;{}_tp_{60} = \frac{70-t}{70}\;\frac{50-t}{50}$$

So P((30) and (60) both survive 20 years)

$$= P(\min(T_x, T_y) > 20) = {}_{20}p_{30:60}$$

$$= \frac{70-20}{70}\;\frac{50-20}{50} = \frac{5}{7}\;\frac{3}{5} = \boxed{\frac{3}{7}}$$

could say $\mu_x = \frac{1}{100-x}$ $(= \mu_{x+0} = \mu_0(x))$

$\mu_y = \frac{1}{110-y}$ $(= \mu_{y+0} = \mu_0(y))$.

64] $\mathring{e}_{xy:\overline{n|}} = \int_0^n {}_tp_{xy}\;dt$

ex] For 2 ind lives (x) and (y)

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You are given $\mu_x = 0.01$ and $\mu_y = 0.02$. Calculate the expected time until both die.

Soln want $\mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}$.

constant mortality so $T_x \sim EXP(.01)$, $T_y \sim EXP(.02)$, $T_{xy} \sim EXP(\mu_x + \mu_y) = EXP(.03)$. so

$$\mathring{e}_{\overline{xy}} = \frac{1}{.01} + \frac{1}{.02} - \frac{1}{.03} = \boxed{116.6667}$$

ex) For 2 ind lives (40) and (55), mortality for (40) follows De Moivre's law with $\omega = 100$. Mortality for (55) follows De Moivre's law with $\omega = 115$. Find the expected amount of time until the last death.

soln) want $\mathring{e}_{\overline{40:55}}$. $100 - 40 = 60 = 115 - 55$

So $T_{xy} \sim GD(2, 60)$ and $\mathring{e}_{40:55} = \frac{\omega - x}{2+1} = \frac{60}{3} = 20$

$\mathring{e}_{40} = \mathring{e}_{55} = \frac{\omega - x}{2} = \frac{60}{2} = 30$

So $\mathring{e}_{\overline{40:55}} = \mathring{e}_{40} + \mathring{e}_{55} - \mathring{e}_{40:55} = 30 + 30 - 20 = \boxed{40}$

ex] suppose ${}_tP_x = \frac{a-t}{a}$, ${}_tP_y = \frac{b-t}{b}$

with $T_x \sim U(0,a) \perp\!\!\!\perp T_y \sim U(0,b)$ where

$a < b$. [In general, $a = \min(\omega_x - x, \omega_y - y)$, $b = \max(\omega_x - x, \omega_y - y)$]

Then $\overset{\circ}{e}_{xy} = \int_0^\infty {}_tP_{xy}\,dt = \int_0^a \frac{a-t}{a}\,\frac{b-t}{b}\,dt$

Since the xy status can't last longer than a.

So $\overset{\circ}{e}_{xy} = \cdots = \frac{a}{2} - \frac{a^2}{6b}$ and

$$\overset{\circ}{e}_{\overline{xy}} = \frac{a}{2} + \frac{b}{2} - \frac{a}{2} + \frac{a^2}{6b} = \frac{b}{2} + \frac{a^2}{6b}.$$

65] ${}_{n|m}q_{\overline{xy}} = {}_nP_{\overline{xy}} - {}_{n+m}P_{\overline{xy}} = P(n < T_{\overline{xy}} \le n+m)$

ex] (40) $\perp\!\!\!\perp$ (55), (40) follows DeMoivre law with $\omega = 100$ and (55) follows a DeMoivre law with $\omega = 105$. Calculate the expected amount of time between the deaths of (40) and (55) (between 1st and last death).

soln] want $\overset{\circ}{e}_{\overline{40:55}} - \overset{\circ}{e}_{40:55}$

$= \overset{\circ}{e}_{40} + \overset{\circ}{e}_{55} - 2\,\overset{\circ}{e}_{40:55}$

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$\overset{\circ}{e}_{40} = \frac{\omega - x}{2} = \frac{100-40}{2} = 30$, $\overset{\circ}{e}_{55} = \frac{105-55}{2} = 25$

$\overset{\circ}{e}_{xy} = \frac{a}{2} - \frac{a^2}{6b}$

$100-40=60=b$

$105-55=50=a$ since $50<60$

$\overset{\circ}{e}_{40:55} = \frac{50}{2} - \frac{(50)^2}{6(60)} = \frac{325}{18}$

secHW3 #1

so $\overset{\circ}{e}_{\overline{40:55}} - \overset{\circ}{e}_{40:55} = 30 + 25 - 2\left(\frac{325}{18}\right)$

$= 18\frac{8}{9} = \boxed{18,8889}$

Back to annuities and insurance

66] ${}_tP_{\overline{xy}} = {}_tP_x + {}_tP_y - {}_tP_{xy}$ and

if $T_x \perp\!\!\!\perp T_y$, ${}_tP_{xy} = {}_tP_x \; {}_tP_y$

since $\min(T_x, T_y) + \max(T_x, T_y) = T_x + T_y = T_{xy} + T_{\overline{xy}}$

so $P(T_{xy} > t) + P(T_{\overline{xy}} > t) = P(T_x > t) + P(T_y > y)$

67] $\bar{A}_{xy} + \bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y$

${}^2\bar{A}_{xy} + {}^2\bar{A}_{\overline{xy}} = {}^2\bar{A}_x + {}^2\bar{A}_y$

$\bar{a}_{xy} + \bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y$

19.5

68] Similar equalities are true for discrete insurances, term insurances, endowments, deferred insurances, temporary annuities and deferred annuities.

(Rarely on actuarial exams)

§12.4.5

69] p307-8 A unit benefit paid at failure of (x) only if (x) fails before (y) (eg (x) parent, (y) child) has

$$APV = \bar{A}^{\ \,1}_{xy} = \int_0^\infty e^{-\delta t} \underbrace{{}_t p_{xy}}_{S_{T_{xy}}(t)} \mu_{x+t}\, dt$$

70] If the unit payment is paid at failure of (x) only if $\overset{\text{child}}{(x)}$ fails after $\overset{\text{parent}}{(y)}$,

the APV is $\bar{A}^{\ \,2}_{xy} = \bar{A}_x - \bar{A}^{\ \,1}_{xy}$

More results related to §12.3

71] ${}_\infty q^{\ \,1}_{xy}$ = P[(x) dies before (y)] =

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$${}_\infty q_{xy}^{\;\;2} = P\left[(y) \text{ dies after } (x)\right].$$

Since either (x) or (y) dies 1st,

$${}_\infty q^{1}_{xy} + {}_\infty q_{xy}^{\;\;1} = 1.$$

Since either (x) or (y) dies 2nd,

$${}_\infty q^{2}_{xy} + {}_\infty q_{xy}^{\;\;2} = 1.$$

Let $T_x \perp\!\!\!\perp T_y$.

72] If $\mu_y(t) = k\,\mu_x(t)$, then P((x) dies 1st)

$$= P\left[(x) \text{ dies before } (y)\right] = \frac{1}{k} P\left[(y) \text{ dies before } (x)\right]$$

$$= \frac{1}{K} P\left[(y) \text{ dies 1st}\right].$$

so ${}_n q^{1}_{xy} = \frac{1}{k}\,{}_n q_{xy}^{\;\;1}$ and

$${}_n q_{xy} = {}_n q^{1}_{xy} + {}_n q_{xy}^{\;\;1} = \underbrace{{}_n q^{1}_{xy} + k\,{}_n q^{1}_{xy}}_{(1+k)\,{}_n q^{1}_{xy}}$$

If joint life status fails in n years then either (x) dies 1st in n years or (y) dies 1st in n years,

so $\boxed{{}_n q^{1}_{xy} = \frac{{}_n q_{xy}}{1+k}}$

73] Application: if $T_x \sim Exp(\mu_x) \perp\!\!\!\perp T_y \sim Exp(\mu_y)$

then $\mu_y = \frac{\mu_y}{\mu_x}\mu_x$ has $k = \frac{\mu_y}{\mu_x}$, so ${}_n q^{1}_{xy} = \dfrac{1-e^{-n(\mu_x+\mu_y)}}{1+\frac{\mu_y}{\mu_x}}$.

So $${}_\infty q^1_{xy} \underset{\substack{\uparrow \\ \text{let } n\to\infty}}{=} \frac{1}{1+\frac{\mu_y}{\mu_x}} = \frac{\mu_x}{\mu_x+\mu_y}.$$

74] Let $T_x \sim U(0, a = \omega_x - x) \perp\!\!\!\perp T_y \sim U(0, b = \omega_y - y)$.

Then $${}_n q^1_{xy} = \begin{cases} \frac{n}{a} - \frac{n^2}{2ab}, & n \le \min(a,b) \\ 1 - \frac{a}{2b}, & n \ge a \text{ and } b \ge a \\ \frac{b}{2a}, & n \ge a \text{ and } b \le a \end{cases}$$

whole life annuity

75]* $\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}$.

If $T_x \sim EXP(\mu_x) \perp\!\!\!\perp T_y \sim EXP(\mu_y)$,

then $\bar{a}_x = \frac{1}{\delta+\mu_x}$, $\bar{a}_y = \frac{1}{\delta+\mu_y}$, $\bar{a}_{xy} = \frac{1}{\delta+\mu_x+\mu_y}$.

*Note: Since $T_{xy} = \min(T_x, T_y)$ is T_x or T_y
and $T_{\overline{xy}} = \max(T_x, T_y)$ is T_y or T_x,
$$g(T_{xy}) + g(T_{\overline{xy}}) = g(T_x) + g(T_y).$$

In fact, if $g(a,b) = g(b,a)$, then $g(T_{xy}, T_{\overline{xy}}) = g(T_x, T_y)$.
See 25]

Do §10.7 later.

endexam material

begin exam 2 material

Ch13 MS82 5/21

Ch 9 1] In multiple decrement models 402 !!

(x) is subject to multiple contingencies.

Each type of failure is called a decrement.

ex] Employee benefit plans pay benefits that differ depending on whether the worker retired, died or became disabled.

2] Multiple decrement theory is the theory of competing risks in biostatistics.

ex] life insurance with 1000 insured who can leave by death = cause 1 or withdrawal = cause 2

3] $q_x^{(j)}$ = P[(x) fails in the next year due to cause j]

q sub x upper j

4] $q_x^{(\tau)}$ = P[(x) fails in the next year] = $\sum_{j=1}^{m} q_x^{(j)}$

q sub x upper τ

$= q_x^{(1)} + \dots + q_x^{(m)}$ if there are m causes

since the distinct causes are disjoint.

5] P((x) does not fail in the next year) =

$p_x^{(\tau)} = 1 - q_x^{(\tau)}$.

p sub x upper τ

6] p336 $d_x^{(j)}$ = (expected) # of people in group at age x (be decremented from the group) who will fail before age x+1 due to cause j.

7] $d_x^{(\tau)} = \sum_{j=1}^{m} d_x^{(j)}$ = (expected) # of people in group at age x who will fail before age x+1.

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8] expected # who fail due to cause j in $(x, x+n] \equiv {}_nd_x^{(j)}$

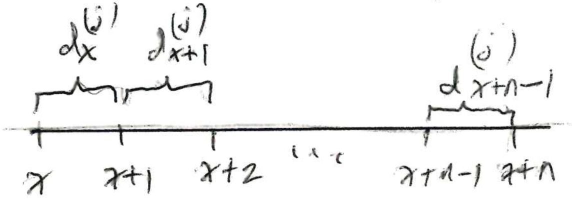
$${}_nd_x^{(T)} = \sum_{j=1}^{m} {}_nd_x^{(j)} = \text{expected \# who fail in } (x, x+n].$$

$${}_nq_x^{(j)} = P\,(\text{of failure due to cause } j \text{ in } (x, x+n])$$

$${}_nq_x^{(T)} = P\,(\text{of failure in } (x, x+n]) = \sum_{j=1}^{m} {}_nq_x^{(j)}$$

$${}_np_x^{(T)} = 1 - {}_nq_x^{(T)} = P\,(\text{of surviving in } (x, x+n])$$

9) $${}_nd_x^{(j)} = \sum_{t=0}^{n-1} d_{x+t}^{(j)}$$



10] p336 $l_x^{(j)}$ = # in group at age x who eventually fail due to cause j

radix (= # at risk)

11] $l_x^{(T)}$ = total # in group at age $x = \sum_{j=1}^{m} l_x^{(j)}$

since everyone eventually fails from one of the m causes.

12] $d_x^{(j)} = l_x^{(T)} q_x^{(j)}$ = (expected) # failing in $(x, x+1]$ due to cause j

$d_x^{(T)} = l_x^{(T)} q_x^{(T)}$ = (expected) # failing in $(x, x+1]$

$${}_nd_x^{(j)} = l_x^{(T)}\,{}_nq_x^{(j)}$$

$${}_nd_x^{(T)} = l_x^{(T)}\,{}_nq_x^{(T)}$$

$$l_{x+1}^{(T)} = l_x^{(T)} - d_x^{(T)}$$

$$d_{x+k}^{(T)} = \overbrace{l_x^{(T)}\,{}_kp_x^{(T)}}^{\text{\# at risk at } x+k}\,\overbrace{q_{x+k}^{(T)}}^{\text{prob of failure in } (x+k,\,x+k+1]} \equiv l_x^{(T)}\,{}_{k|}q_x^{(T)}$$

$$d_{x+k}^{(j)} = l_x^{(T)}\,{}_kp_x^{(T)}\,q_{x+k}^{(j)} \equiv l_x^{(T)}\,{}_nq_x^{(j)}$$

13] $$l_{x+n}^{(T)} = l_x^{(T)}\,{}_np_x^{(T)} \qquad \text{so} \qquad {}_np_x^{(T)} = \frac{l_{x+n}^{(T)}}{l_x^{(T)}}$$

more table formulas

14] p337 multiple-decrement table:

radix = initial # in group = R ($= l_x^{(T)}$ for smallest x in table)

$l_x^{(T)}$ = # left in group at age x

m=2: double decrement table, m=3: triple decrement table