15) Know

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x	$q_x^{(1)}$	$\cdots$	$q_x^{(m)}$
45	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$
50			

Given $l_{45}^{(T)}$ and the above table, be able to find

$$q_x^{(T)} = \sum_{i=1}^{m} q_x^{(j)}, \quad p_x^{(T)} = 1 - q_x^{(T)}, \quad l_x^{(T)} = p_{x-1}^{(T)}\, l_{x-1}^{(T)}$$

$$d_x^{(j)} = l_x^{(T)}\, q_x^{(j)}$$

ex} p337 initial group size = 1000 $\qquad l_{45}^{(T)} = 1000 \qquad$ radix is given

given

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x = \sum_{j=1}^{2} q_x^{(j)}$	$p_x^{(T)} = 1 - q_x^{(T)}$	$l_x^{(T)}$	$d_x^{(1)} = l_x^{(T)} q_x^{(1)}$	$d_x^{(2)} = l_x^{(T)} q_x^{(2)}$
45	.011	.1	.111	.889	1000	11.00 = 1000(.011)	100 = 1000(.1)
46	.012	.1	.112	.888	889 = 1000(.899)	10.67 = 889(.012)	88.9 = 889 (.1
47	.013	.1	.113	.887	789.43	10.26	78.94 = 789.43(.1
48	.014	.1	.114	.886	700.23	9.80	70.02 = 700.23 (.1
49	.015	.1	.115	.885	620.40	9.31	62.04 = 620.40(.1
50	.016	.1	.116	.884	549.05 = .885(620.40)	8.78 = 549.05(.016)	54.91 = 549.05(.1)

$l_{x+1}^{(T)} = l_x^{(T)} - d_x^{(1)} - d_x^{(2)}$ except for rounding

16} Know Then many quantities can be found as in ch 3.

a) $\boxed{{}_n p_x^{(T)} = \dfrac{l_{x+n}^{(T)}}{l_x^{(T)}}}$ so ${}_3 p_{46}^{(T)} = \dfrac{l_{49}^{(T)}}{l_{46}^{(T)}} = \dfrac{620.40}{889.00} = 0.6979$

$$= p_{46}^{(T)}\, p_{47}^{(T)}\, p_{48}^{(T)} = \frac{l_{47}^{(T)}}{l_{46}^{(T)}}\, \frac{l_{48}^{(T)}}{l_{47}^{(T)}}\, \frac{l_{49}^{(T)}}{l_{48}^{(T)}} \qquad \left(\text{"} = P(T_{46} > 3)\text{"}\right)$$

b) $\boxed{{}_n d_x^{(j)} = \sum_{t=0}^{n-1} d_{x+t}^{(j)}}$

so ${}_2 d_{47}^{(2)}$ = expected # who die (become inactive) in $(47, 47+2]$ from cause 2

$$= d_{47}^{(2)} + d_{48}^{(2)} = 78.94 + 70.02 = 148.96$$

$\boxed{{}_n q_x^{(j)} = {}_n d_x^{(j)} / l_x^{(\tau)}}$

22.5

c) ${}_2 q_{46}^{(1)} = P(X \le 46+2$ and dies from cause 1 $\mid X > 46)$
becomes inactive

$= \dfrac{{}_2 d_{46}^{(1)}}{l_{46}^{(\tau)}} = \dfrac{d_{46}^{(1)} + d_{47}^{(1)}}{l_{46}^{(\tau)}} = \dfrac{10.67 + 10.26}{889.00} = 0.02354$

$= q_{46}^{(1)} + p_{46}^{(\tau)} q_{47}^{(1)} = P(X \le 47$ and dies from cause 1 $\mid X > 46)$

$+ \underbrace{P(\text{"}T_{46} > 1\text{"})}_{P(X>47 \mid X>46)} P(X \le 48$ and dies from cause 1 $\mid X > 47)$

$\boxed{{}_{n|m} q_x^{(j)} = \dfrac{\sum_{t=0}^{m-1} d_{x+n+t}^{(j)}}{l_x^{(\tau)}}}$

d) ${}_{2|2} q_{45}^{(1)} = P(\text{"}2 < T_{45} \le 2+2\text{"}$ and death is due to cause 1)

$= \dfrac{d_{47}^{(1)} + d_{48}^{(1)}}{l_{45}^{(\tau)}} = \dfrac{10.26 + 9.80}{1000} = .02006$

$= \dfrac{\text{expected \# to die of cause 1 in } (46,47] \cup (47,48]}{\text{expected \# alive at } 45}$

$\boxed{{}_{n|} q_x^{(\tau)} = {}_{n|1} q_x^{(\tau)} = d_{x+n}^{(\tau)} / l_x^{(\tau)}}$

e) ${}_{2|} q_{46}^{(\tau)} = P(\text{"}2 < T_{46} \le 3\text{"}) = \dfrac{d_{48}^{(1)} + d_{48}^{(2)}}{l_{46}^{(\tau)}} = \dfrac{d_{48}^{(\tau)}}{l_{46}^{(\tau)}}$

$= \dfrac{\text{expected \# who die in } (48,49]}{\text{expected \# alive at } 46} = \dfrac{9.80 + 70.20}{889.0} = .08979$

$= p_{46}^{(\tau)} p_{47}^{(\tau)} q_{48}^{(\tau)} = \dfrac{l_{47}^{(\tau)}}{l_{46}^{(\tau)}} \dfrac{l_{48}^{(\tau)}}{l_{47}^{(\tau)}} \dfrac{d_{48}^{(\tau)}}{l_{48}^{(\tau)}}$

or ${}_{2|} q_{46}^{(\tau)} = {}_2 p_{46}^{(\tau)} - {}_3 p_{46}^{(\tau)} = \dfrac{l_{48}^{(\tau)}}{l_{46}^{(\tau)}} - \dfrac{l_{49}^{(\tau)}}{l_{46}^{(\tau)}} =$

$\dfrac{700.23}{889.00} - .6979 = .08976$

Go to 34

17] $K_x = k$ means (x) fails in the $(k+1)$th interval
$J_x = j$ means failure was due to the jth cause
$k = 0, 1, 2, \ldots$ $j = 1, \ldots, m$.

18] p339 The joint prob function of K_x and J_x

is $p_{K_x, J_x}(k, j) = P(K_x = k, J_x = j) = {}_{k|}q_x^{(j)} = \dfrac{d_{x+k}^{(j)}}{\ell_x^{(T)}}$

ex] $P((x)$ fails between $\underbrace{x+4 \text{ and } x+5}_{K_x = 4}$ due to cause 1$)$
$= {}_{4|}q_x^{(1)}$

19] The marginal prob function of K_x is

$$p_{K_x}(k) = P(K_x = k) = \sum_{j=1}^{m} p_{K_x, J_x}(k, j) =$$

$${}_{k|}q_x^{(T)} = \frac{\sum_{j=1}^{m} d_{x+k}^{(j)}}{\ell_x^{(T)}} = \frac{\text{sum of } d^{(j)} \text{ values in row of multiple decrement table corresponding to age } x+k}{\ell_x^{(T)}}$$

$= P[(x)$ fails in $(k+1)$th interval (year) due to any cause$)$.

20] The marginal prob fn of J_x is $P(J_x = j) =$

$$p_{J_x}(j) = \sum_{k=0}^{\infty} p_{K_x, J_x}(k, j) = \frac{\sum_{k=0}^{\infty} d_{x+k}^{(j)}}{\ell_x^{(T)}} = P[(x) \text{ will}$$

eventually fail due to cause j] $=$

$$\frac{\text{sum of } d^{(j)} \text{ values in the col. of the multiple decrement table corresponding to cause } j}{\ell_x^{(T)}}$$

ex] p340 For the table above 16], find a) $P_{K_x, J_x}(2,1)$ and b) $P_{K_x}(3)$ for a person of age 46. (23.5

soln] a) $P_{K_x, J_x}(2,1) = {}_{K|}q_x^{(j)} = {}_{2|}q_{46}^{(1)} = \dfrac{d_{x+k}^{(j)}}{l_x^{(\tau)}} = \dfrac{d_{48}^{(1)}}{l_{46}^{(\tau)}} = \dfrac{9.80}{889.0} = 0.01102$

b) $P_{K_x}(K) = {}_{3|}q_x^{(\tau)} = {}_{3|}q_{46}^{(\tau)} = \dfrac{\sum_{j=1}^{m} d_{x+k}^{(j)}}{l_x^{(\tau)}} = \dfrac{d_{49}^{(1)} + d_{49}^{(2)}}{l_{46}^{(\tau)}}$

$= \dfrac{9.31 + 62.04}{889.0} = 0.08026$

21] ${}_np_x^{(\tau)} = p_x^{(\tau)} p_{x+1}^{(\tau)} p_{x+2}^{(\tau)} \cdots p_{x+n-1}^{(\tau)}$ since LHS $= \dfrac{l_{x+n}^{(\tau)}}{l_x^{(\tau)}}$ = RHS $= \dfrac{l_{x+1}^{(\tau)}}{l_x^{(\tau)}} \dfrac{l_{x+2}^{(\tau)}}{l_{x+1}^{(\tau)}} \dfrac{l_{x+3}^{(\tau)}}{l_{x+2}^{(\tau)}} \cdots \dfrac{l_{x+n}^{(\tau)}}{l_{x+n-1}^{(\tau)}}$

p13.2 ex] Suppose failure is leaving a company either by death (cause1) or retirement (cause2) of 1000 64 year olds: $q_{64}^{(1)} = 0.02$ and $q_{64}^{(2)} = 0.30$. These probabilities refer to the fractions of employees expected to die and retire during the next year given the employees are exposed to both decrements at the same time. Expect 20 deaths and 300 retirements. But if rules were changed so that no one could retire until 65, we would expect more than 20 deaths since some of the 300 who would have retired will die.

22] p341 The higher probability of death in the absence of other decrements is $q_x'^{(1)} =$

absolute rate of death.

23] The absolute rate of decrement due to cause j over $(x, x+n]$ = ${}_n q_x^{\prime(j)}$ = P [failing due to cause j in $(x, x+n]$] if no other causes of decrement were acting.

${}_1 q_x^{\prime(j)} = q_x^{\prime(j)}$. primes go with single decrement probabilities

24] If $m=1$ (single decrement case of ch 2-6 M401), then $q_x^{\prime(j)} = q_x$ and $q_x^{\prime(j)}$ is the probability of decrement due to cause j in the associated single decrement table.

25] ${}_n q_x^{\prime(j)} \geq {}_n q_x^{(j)}$

no primes on $T_x^{(j)}$ or $\mu_{x+t}^{(j)}$

§13.3 26] p 341 Let $T_x^{(j)}$ = time to failure RV = RV for time until (x) fails due to cause j in the absense of other decrements.

(so $T_x^{(j)}$ is like T_x in ch 2-6). M401

27] i) CDF $F_{T_x^{(j)}}(t) = P(T_x^{(j)} \leq t) = {}_t q_x^{\prime(j)} = \int_0^t f_{T_x^{(j)}}(s)\,ds$
$= \int_0^t {}_s p_x^{(j)} \mu_{x+s}^{(j)}\,ds$

ii) survival fn $S_{T_x^{(j)}}(t) = P(T_x^{(j)} > t) = 1 - {}_t q_x^{\prime(j)} = {}_t p_x^{\prime(j)}$
$= \exp\left[-\int_0^t \mu_{x+s}^{(j)}\,ds\right]$.

iii) Force of failure (mortality) (use failure since not all causes are death) (24.5)

$\mu_{T_x^{(j)}}(t) = \mu_{x+t}^{(j)}$ (no prime) $\left(= \mu_{x+t}^{\prime(j)} \right.$. The force is instantaneous so it is the same whether operating in the single decrement or multiple decrement environment. By convention, $\mu_{x+t}^{\prime(j)}$ is not used.)

$$\mu_{x+t}^{(j)} = \frac{-\frac{d}{dt} S_{T_x^{(j)}}(t)}{S_{T_x^{(j)}}(t)} = \frac{-\frac{d}{dt}\, {}_tp_x^{\prime(j)}}{{}_tp_x^{\prime(j)}}$$

iv) pdf $\quad f_{T_x^{(j)}}(t) = S_{T_x^{(j)}}(t)\, \mu_{T_x^{(j)}}(t) = {}_tp_x^{\prime(j)}\, \mu_{x+t}^{(j)}$

28] i) survival function ${}_tp_x^{(T)} = 1 - {}_tq_x^{(T)} = \exp\left[-\int_0^t \mu_{x+s}^{(T)}\, ds\right]$
$= \prod_{j=1}^{m} {}_tp_x^{\prime(j)}$

ii) cdf $\quad {}_tq_x^{(T)} = 1 - {}_tp_x^{(T)} = \int_0^t {}_sp_x^{(T)}\, \mu_{x+s}^{(T)}\, ds$

iii) force of failure $\quad \mu_{x+t}^{(T)} = \dfrac{-\frac{d}{dt}\, {}_tp_x^{(T)}}{{}_tp_x^{(T)}} = \sum_{j=1}^{m} \mu_{x+t}^{(j)}$

29] some books derive ${}_tp_x^{(T)} = \prod_{j=1}^{m} {}_tp_x^{\prime(j)}$ and

$\mu_{x+t}^{(T)} = \sum_{j=1}^{m} \mu_{x+t}^{(j)}$ assuming causes are independent

but exercise 13.7 shows the formulas work even if the causes are not independent.

p.344

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TS.

30] ${}_{t}q_x^{(j)} = \int_0^t {}_{s}p_x^{(\tau)} \mu_{x+s}^{(j)} \, ds$

and $\mu_{x+t}^{(j)} = \dfrac{\frac{d}{dt} \, {}_{t}q_x^{(j)}}{{}_{t}p_x^{(\tau)}}$

31] ${}_{\infty}q_x^{(j)} = \lim\limits_{t \to \infty} {}_{t}q_x^{(j)}$

*32] Given $\mu_{x+t}^{(1)}$ and $\mu_{x+t}^{(2)}$, find

i) $\mu_{x+t}^{(\tau)} = \mu_{x+t}^{(1)} + \mu_{x+t}^{(2)}$

ii) ${}_{t}p_x^{(\tau)} = \exp\left[-\int_0^t \mu_{x+s} \, ds\right] = 1 - {}_{t}q_x^{(\tau)}$

iii) ${}_{t}q_x^{(j)} = \int_0^t {}_{s}p_x^{(\tau)} \mu_{x+s}^{(j)} \, ds$

iv) ${}_{\infty}q_x^{(j)} = \lim\limits_{t \to \infty} {}_{t}q_x^{(j)}$

*ex] 13.4] p344 $\mu_{x+t}^{(1)} = 0.1$ $\mu_{x+t}^{(2)} = 0.2$ $\forall t$

a) $\mu_{x+t}^{(\tau)} = \mu_{x+t}^{(1)} + \mu_{x+t}^{(2)} = 0.30$

Let W be the RV corresponding to $\mu_{x+t}^{(\tau)}$. So $W \sim EXP(.3)$.

b) $S_W(t) = e^{-\mu t} = e^{-0.3t} = {}_{t}p_x^{(\tau)} = \exp\left[-\int_0^t .3 \, ds\right]$

$= \exp\left(-.3s\Big|_0^t\right) = \exp\left[-.3t - 0\right] = e^{-.3t} = {}_{t}p_x^{(\tau)}$

c) ${}_{t}q_x'^{(1)} = 1 - {}_{t}p_x'^{(1)} = 1 - \exp\left[-\int_0^t .1 \, ds\right] = 1 - \exp\left[-.1s\Big|_0^t\right]$

$= 1 - e^{-.1t}$ as expected since $T_x^{(1)} \sim EXP(.1)$.

d) ${}_t q_x^{(1)} = P$ (of failure due to cause 1 in $(x, x+t]$)

(255)

$$= \int_0^t {}_s p_x^{(\tau)} \, \mu_{x+s}^{(1)} \, ds = \int_0^t e^{-.3s} \, .1 \, ds$$

$$= .1 \left(\frac{e^{-.3s}}{-.3} \Big|_0^t \right) = -\frac{1}{3}\left(e^{-.3t} - 1\right) = \frac{1}{3}\left(1 - e^{-.3t}\right)$$

e) ${}_\infty q_x^{(1)} = \lim_{t \to \infty} \frac{1}{3}\left(1 - e^{-.3t}\right) = \frac{1}{3}$

33] a) ${}_t q_x^{(\tau)} = \sum_{j=1}^{m} {}_t q_x^{(j)} =$

P [(x) fails in next t yeaars] — between x and x+t

b) ${}_{t|} q_x^{(j)} = {}_t p_x^{(\tau)} \, q_{x+t}^{(j)} =$

P[(x) fails between x+t and x+t+1 due to cause j]

c) ${}_{t|m} q_x^{(j)} = \sum_{k=t}^{t+m-1} {}_k p_x^{(\tau)} \, q_{x+k}^{(j)} =$

P[(x) fails between x+t and x+t+m due to cause j]

see 16]

*34] The illustrative service table is needed for some problems. See handout.

*ex] Using the illustrative service table, (handout) 402 #5 (26)

calculate the probability that a life at age 40 retires before age 65.

soln] Decrements are d for death, w for withdrawal, i for disability, and r for retirement.

The prob is the ratio of the total number of retirements up to age 65 over the # of lives at age 40

$$ {}_{25}q_{40}^{(r)} = \frac{\sum_{k=0}^{24} d_{40+k}^{(r)}}{l_{40}^{(T)}} $$

$d_{64}^{(r)}$ is # retirements from 64 to 65

x	$d_x^{(r)}$
60	3552
61	1587
62	2692
63	1350
64	2006

$$ = \frac{3552 + 1587 + 2692 + 1350 + 2006}{36943} $$

$d_{40}^{(r)} = \dots = d_{59}^{(r)} = 0$

$$ = \boxed{0.30282} $$

35] The $l_x^{(T)}$ are sometimes called "active lives". so nonactive lives (died, retired, withdrew or had a disability). So prob a policy holder age 60 will be active after 2 years is

$$ {}_2p_{60}^{(T)} = \frac{l_{62}^{(T)}}{l_{60}^{(T)}} = \frac{\text{\#active after 62}}{\text{\#active after 60}} = \frac{18106}{23856} = \boxed{.7590} $$

illust service table

36] *All single decrement formulas apply (265)
to functions with (τ), but such
formulas may not apply with (j).

ex] ${}_tp_x^{(\tau)} = \exp\left[-\int_0^t \mu_{x+s}^{(\tau)}\, ds\right]$,

but ${}_tp_x^{(j)} \neq \exp\left[-\int_0^t \mu_{x+s}^{(j)}\, ds\right]$.

p357 (problem 13.4)

37 *The probability that someone
succumbed to decrement j at time t,
given that the person $\underbrace{\text{succumbed}}_{\text{became inactive}}$ to some decrement at
time t

$$= \frac{\mu_{x+t}^{(j)}}{\mu_{x+t}^{(\tau)}}.$$

ex] In a triple decrement model,

$\mu_x^{(1)}(t) = \frac{1}{50-t}$ $\quad 0 < t < 50$

$\mu_x^{(2)}(t) = .01$ $\quad t > 0$

$\mu_x^{(3)}(t) = 0.0015t$ $\quad t > 0$.

Suppose (x) succumbs at time $t = 10$.
Calculate the prob that the person