succumbed due to decrement 1. 402 17

(27)

Soln] $\mu_x^{(1)}(10) = \frac{1}{40} = 0.025$

$\mu_x^{(2)}(10) = 0.01$

$\mu_x^{(3)}(10) = 0.0015(10) = 0.015$

and $\mu_{x(10)}^{(T)} = \sum_{j=1}^{3} \mu_x^{(j)}(10) = 0.05$

So $\text{Prob} = \frac{\mu_x^{(1)}(10)}{\mu_x^{(\tau)}(10)} = \frac{.025}{.05} = \boxed{0.5}$

38] To go from multiple to single decrement quantities

$\uparrow$ primes

i) calculate ${}_tp_x^{(\tau)} = 1 - {}_tq_x^{(T)} = 1 - \sum_{j=1}^{m} {}_tq_x^{(j)}$

ii) calculate $\mu_{x+t}^{(j)} = \frac{\frac{d}{dt}\, {}_tq_x^{(j)}}{{}_tp_x^{(T)}}$

iii) calculate ${}_tp_x'^{(j)} = \exp\left[-\int_0^t \mu_{x+s}^{(j)}\, ds\right]$

As a check, ${}_tp_x^{(\tau)} = \prod_{j=1}^{m} {}_tp_x'^{(j)}$

ex] $m=3$, $\quad {}_tq_x^{(1)} = 0.5\left[1-e^{-0.04t}\right]$ 22.5

$${}_tq_x^{(2)} = 0.4\left[1-e^{-0.04t}\right]$$

$${}_tq_x^{(3)} = 0.1\left[1-e^{-0.04t}\right]$$

Find ${}_tp_x'^{(j)}$ for $j=1,2,3$.

Soln] $${}_tp_x^{(\tau)} = 1-\sum_{j=1}^{m} {}_tq_x^{(j)} = 1-\left[1-e^{-.04t}\right] = e^{-.04t}$$

$$\mu_{x+t}^{(1)} = \frac{\frac{d}{dt}\,{}_tq_x^{(1)}}{{}_tp_x^{(\tau)}} = \frac{.5(.04)\,e^{-.04t}}{e^{-.04t}} = .02$$

$$\mu_{x+t}^{(2)} = \frac{\frac{d}{dt}\,{}_tq_x^{(2)}}{{}_tp_x^{(\tau)}} = .4(.04) = .016$$

$$\mu_{x+t}^{(3)} = \frac{\frac{d}{dt}\,{}_tq_x^{(3)}}{{}_tp_x^{(\tau)}} = .1(.04) = .004$$

EXP

So $${}_tp_x'^{(1)} = e^{-0.02t}$$

$${}_tp_x'^{(2)} = e^{-.016t}$$

$${}_tp_x'^{(3)} = e^{-.004t}$$

EXP survival functions

39] To go from single to multiple decrement quantities

i) calculate $\mu_{x+t}^{(j)} = -\frac{d}{dt}\log\left({}_tp_x'^{(j)}\right) =$

$$= \frac{-\frac{d}{dt}\, {}_tp_x'^{(j)}}{{}_tp_x'^{(j)}} = \frac{\frac{d}{dt}\, {}_tq_x'^{(j)}}{{}_tp_x'^{(j)}}$$

ii) Get ${}_tp_x^{(\tau)} = \prod_{j=1}^{m} {}_tp_x'^{(j)} = \exp\left[-\int_0^t \mu_{x+s}^{(\tau)}\, ds\right]$

where $\mu_{x+t}^{(\tau)} = \sum_{j=1}^{m} \mu_{x+t}^{(j)}$.

iii) ${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\, \mu_{x+s}^{(j)}\, ds$

ex) $m=2$ ${}_tp_x'^{(1)} = \frac{50-t}{50}$ $t<50$ DeMoivre(50)

${}_tp_x'^{(2)} = \left(\frac{60-t}{60}\right)^2$ $t<60$ GD(2,60)

Find ${}_tq_x^{(1)}$ for $t<50$

soln) $\mu_{x+t}^{(1)} = \frac{1}{50-t}$, $\mu_{x+t}^{(2)} = \frac{2}{60-t}$

(see El review 63)

$${}_tp_x^{(\tau)} = \prod_{j=1}^{m} {}_tp_x'^{(j)} = \frac{50-t}{50}\left(\frac{60-t}{60}\right)^2$$

$${}_tq_x^{(1)} = \int_0^t {}_sp_x^{(\tau)}\, \mu_{x+s}^{(1)}\, ds =$$

$$\int_0^t \frac{50-s}{50}\left(\frac{60-s}{60}\right)^2 \frac{1}{50-s}\, ds = \int_0^t \frac{1}{(60)^2(50)} (60-s)^2\, ds$$

$$= \frac{-1}{(60)^2(50)} \left. \frac{(60-s)^3}{3} \right|_0^t = \frac{-1}{(60)^2(50)} \left[\frac{(60-t)^3 - (60)^3}{3} \right]$$

$$= \frac{(60)^3 - (60-t)^3}{3(60)^2(50)}$$

§10.7

1) often (x) and (y) are dependent and are exposed to a common hazard, called a common shock.

ex) married people who travel together

2) Let μ^*_{x+t} be the force of failure at time t that are specific to (x) but not (y). Similarly, let μ^*_{y+t} be the force of failure at time t that are specific to (y) but not (x).

Let the common hazard = common shock for both (x) and (y) be constant

$\mu^c_t \equiv \lambda, \quad t \geq 0.$

3) * p320-1 The common shock model assumes

$$\mu_{x+t} = \mu^*_{x+t} + \lambda$$
$$\mu_{y+t} = \mu^*_{y+t} + \lambda, \quad \text{and}$$

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the total force of failure for the joint status (xy) is

$$\mu_{x+t:y+t} = \mu^*_{x+t} + \mu^*_{y+t} + \lambda$$
$$(= \mu_{x+t} + \mu_{y+t} - \lambda).$$

4) The 3 forces are additive because they represent hazard factors that are disjoint from each other. There is only one λ since failure of either (x) or (y) due to the common shock constitutes failure of the joint life status.

5) p321

$${}_tp^*_x = \exp\left[-\int_0^t (\mu^*_{x+r} + \lambda)\,dr\right] = e^{-\lambda t} e^{-\int_0^t \mu^*_{x+r}\,dr}$$
$$= {}_tp^*_x\, e^{-\lambda t}$$

$${}_tp_y = \exp\left[-\int_0^t (\mu^*_{y+r} + \lambda)\,dr\right] = e^{-\lambda t} e^{-\int_0^t \mu^*_{y+r}\,dr} =$$
$${}_tp^*_y\, e^{-\lambda t}$$

$${}_tp_{xy} = \exp\left[-\int_0^t (\mu^*_{x+r} + \mu^*_{y+r} + \lambda)\,dr\right] =$$

$$e^{-\int_0^t \mu^*_{x+r}dr}\, e^{-\int_0^t \mu^*_{y+r}dr} \left(e^{-\lambda t}\right) = {}_tp^*_x\, {}_tp^*_y\, e^{-\lambda t} \overbrace{\left(e^{-\lambda t} e^{\lambda t}\right)}^{1}$$

joint life survival function

$$= {}_tp_x\, {}_tp_y\, e^{\lambda t}.$$

6) * p321 An equivalent way to develop the common shock model is to let T_x^* and T_y^* denote the future lifetime random variables for (x) and (y) without regard for the common shock hazard functions. Let W denote the future lifetime of either (x) or (y) with regard to the common shock hazard factors only. Assume $T_x^* \perp\!\!\!\perp T_y^* \perp\!\!\!\perp W$ with $W \sim EXP(\lambda)$.

30.5

$$T_x = \min(T_x^*, W), \quad T_y = \min(T_y^*, W)$$

and $T_x \cancel{\perp\!\!\!\perp} T_y$.

Since $T_x \perp\!\!\!\perp W$, ${}_tP_x = S_{T_x}(t) = S_{T_x^*}(t)\, S_W(t)$

see top of 5) $\quad = {}_tP_x^* e^{-\lambda t}.$

Since $T_y \perp\!\!\!\perp W$, ${}_tP_y = S_{T_y}(t) = S_{T_y^*}(t)\, S_W(t) =$

$${}_tP_y^* e^{-\lambda t}.$$

Note that ${}_tp_x = P(\min(T_x^*, W) > t) = P(T_x^* > t \text{ and } W > t)$

$= P(T_x^* > t)\, P(W > t)$ since $T_x^* \perp\!\!\!\perp W$.

${}_tp_{xy} = P(T_x^* > t,\ T_y^* > t, \text{ and } W > t) =$

${}_tp_x^*\ {}_tp_y^*\ e^{-\lambda t} = {}_tp_x\ {}_tp_y\ e^{\lambda t}$

7) $P(T_x = T_y) = \int_0^\infty {}_tp_{xy}\ \lambda\, dt$

8) If $\lambda = 0$, $T_x^* = T_x \perp\!\!\!\perp T_y = T_y^*$.

$T_{xy} = \min(T_x, T_y)$, $T_{\overline{xy}} = \max(T_x, T_y)$

9) Know Earlier chapter 10 formulas still hold for the common shock model, but T_x is not independent of T_y. For the last survivor status $(\overline{xy})$,

i) $T_{xy} + T_{\overline{xy}} = T_x + T_y$ so

$T_{\overline{xy}} = T_x + T_y - T_{xy}$. If $g(a,b) = g(b,a)$, then $g(T_{xy}, T_{\overline{xy}}) = g(T_x, T_y)$.

ii) survival function $S_{T_{\overline{xy}}}(t) = {}_tp_{\overline{xy}} = P(T_{\overline{xy}} > t) =$

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$_tp_x + {}_tp_y - {}_tp_{xy}$

iii) cdf $F_{T_{\overline{xy}}}(t) = {}_tq_{\overline{xy}} = 1 - S_{T_{\overline{xy}}}(t) = 1 - {}_tp_{\overline{xy}}$.

iv) $\mathring{e}_{\overline{xy}} = E(T_{\overline{xy}}) = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}$.

v) $\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$

vi) $\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}$

p322

10) Know Suppose $T_x^* \sim EXP(\mu_x^*)$, $T_y^* \sim EXP(\mu_y^*)$ $(W \sim EXP(\lambda))$ in the common shock model.

Then $T_x \sim EXP(\underbrace{\mu_x^* + \lambda}_{\mu_x})$, $T_y \sim EXP(\underbrace{\mu_y^* + \lambda}_{\mu_y})$

$T_{xy} \sim EXP(\underbrace{\mu_x^* + \mu_y^* + \lambda}_{\mu_x + \mu_y - \lambda})$ since $\mu_{xy}(t) = \mu_x^* + \mu_y^* + \lambda$.

$\bar{A}_x = \dfrac{\mu_x^* + \lambda}{\mu_x^* + \lambda + \delta}$, $\bar{A}_y = \dfrac{\mu_y^* + \lambda}{\mu_y^* + \lambda + \delta}$, $\bar{A}_{xy} = \dfrac{\mu_x^* + \mu_y^* + \lambda}{\mu_x^* + \mu_y^* + \lambda + \delta}$

$\bar{a}_x = \dfrac{1}{\mu_x^* + \lambda + \delta}$, $\bar{a}_y = \dfrac{1}{\mu_y^* + \lambda + \delta}$, $\bar{a}_{xy} = \dfrac{1}{\mu_x^* + \mu_y^* + \lambda + \delta}$.

and $P(T_x = T_y) = \dfrac{\lambda}{\mu_x^* + \mu_y^* + \lambda}$.

11] * In problems, see whether

μ_x $(= \mu_x^* + \lambda)$, μ_y $(= \mu_y^* + \lambda)$ and λ are

given <u>or</u> are μ_x^*, μ_y^* and λ given.

For HW5 #4 $\mu_x = \mu_y = 0.06$ and $\lambda = 0.02$

are given. So $\mu_x^* = \mu_y^* = 0.04$.

i) If you are told the force of mortality for $(x) = U$, the force of mortality for $(y) = V$, and <u>common shock is incorporated into</u> the forces of mortality U and V, then $U = \mu_{x+t}$ and $V = \mu_{y+t}$.

ii) If you are told the <u>noncommon forces of mortality</u> for (x) and (y) are U and V, then $U = \mu_{x+t}^*$ and $V = \mu_{y+t}^*$.

12} In 7] $P(T_x = T_y) = \int_0^\infty {}_tP_{xy} \, \lambda \, dt$.

$= \lambda \int_0^\infty e^{-t(\mu_x^* + \mu_y^* + \lambda)} dt = \dfrac{\lambda}{\mu_x^* + \mu_y^* + \lambda}$.

end 6107 material

32.5

Ch18 Estimating Survival Models

Overlap with Math 473, 401, 403–404.

1] $S(t) = P(T > t)$.

2] p667 Let $T_1, \ldots, T_n$ be independent and identically distributed (iid).

Let $t_1 \le t_2 \le t_3 \le \cdots \le t_n$ be the observed ordered T_i.

The empirical estimator

$$* \quad \hat{S}_E(t) = \frac{1}{n} \sum_{i=1}^{n} I(T_i > t) = \frac{\# \; T_i > t}{n} = \frac{n_t}{n}$$

where $I(T_i > t) = \begin{cases} 1 & T_i > t \\ 0 & T_i \le t. \end{cases}$

3] p667 Let $N_t = \sum_{i=1}^{n} I(T_i > t)$. So n_t is the observed value of N_t.

$N_t \sim \text{bin}(n, S(t))$.

$$E(\hat{S}_E(t)) = E\left(\frac{N_t}{n}\right) = n\,\frac{S(t)}{n} = S(t).$$

$$V(\hat{S}_E(t)) = \frac{1}{n^2} V(N_t) = \frac{n}{n^2} S(t)(1-S(t)) = \frac{S(t)(1-S(t))}{n}.$$

33.5

$$\hat{V}\left(\hat{S}_E(t)\right) = \frac{\hat{S}_E(t)\left(1-\hat{S}_E(t)\right)}{n}$$

$$= \frac{\frac{n_t}{n}\left(1-\frac{n_t}{n}\right)}{n} = \frac{n_t\,(n-n_t)}{n^3} \qquad \underbrace{\frac{n_t}{n_t}}_{1}$$

$$= \left(\frac{n_t}{n}\right)^2 \left(\frac{n-n_t}{n\,n_t}\right) = \left[\hat{S}_E(t)\right]^2 \left(\frac{1}{n_t} - \frac{1}{n}\right)$$

$$* \boxed{SE\left(\hat{S}_E(t)\right) = \hat{S}_E(t)\sqrt{\frac{1}{n_t} - \frac{1}{n}}} = \sqrt{\hat{V}\left(\hat{S}_E(t)\right)}.$$

4) * A linear or Wald 95% confidence interval CI for θ is $\hat{\theta} \pm 1.96\, SE(\hat{\theta})$.

ex) $\hat{\mu} = 762.76$, $SE(\hat{\mu}) = 217.32$

A 95% CI for μ is

$\hat{\mu} \pm 1.96\, SE(\hat{\mu}) = 762.76 \pm 1.96\,(217.32)$

$= 762.76 \pm 425.9472 = \boxed{[336.8128,\ 1188.7072]}$

5) Know for exam 2!

Suppose 1000 white 71 year old females buy a $t=1$ year \$100000 life insurance policy. Actuaries use $1 - \frac{S(t+x)}{S(x)} = 1 - P(Y > t+x \mid Y > x)$ to estimate how many claims will be filed. Hence actuaries want (for $t=1$) $\frac{S(72)}{S(71)}$. If $\hat{S}(72) = 0.85$ and $\hat{S}(71) = 0.87$, about how many of the 1000 claims will be filed?

Soln) $\left(1 - \frac{.85}{.87}\right) 1000 = 0.02299\,(1000)$

$= \boxed{22.99}$ with $x = 71$ and $t = 1$.

Note t = term of the term insurance, while x = age of insured at purchase.

34.5

6] Know for exam 2

The following 20 survival times are listed from smallest to largest.

15, 22, 38, 49, 62, 71, 91, 102, 131, 145, 177, 198, 247, 279, 319, 359, 469, 526, 703, 790

a) Find $\hat{S}_E(247)$.

b) Find a 95% CI for $\overbrace{S(247)}^{\theta}$

soln) a) $\hat{S}_E(247) = \frac{n_t}{n} = \frac{\# > 247}{n}$

$= \frac{7}{20} = \boxed{0.35}$

b) $SE(\hat{S}_E(t)) = \hat{S}_E(t)\sqrt{\frac{1}{n_t} - \frac{1}{n}}$

$= 0.35\sqrt{\frac{1}{7} - \frac{1}{20}} = 0.1067$

$CI = \hat{S}_E(t) \pm 1.96\, SE(\hat{S}_E(t))$

$= 0.35 \pm 1.96(0.1067) = 0.35 \pm .2090$

$= [0.1410 \ , .5590]$.

Note: if the CI is $[L, U]$

$[\max(0, L), \min(U, 1)]$ is a better CI

since $0 \le \underbrace{S(t)}_{\text{prob}} \le 1$.

7)* Suppose we observe n lives from exact age x to x_K and we have deaths for the K intervals:

interval		$\overbrace{\text{interval length}}^{\text{not given}}$	deaths
1	$[x, x_1)$	L_1	d_1
2	$[x_1, x_2)$	L_2	d_2
$\vdots$		$\vdots$	$\vdots$
K	$[x_{K-1}, x_K)$	L_K	d_K

$\hat{S}_x(0) = 1$

$\hat{S}_x(L_1) = \dfrac{n - d_1}{n} = \hat{S}_x(0) - \dfrac{d_1}{n}$

$$\hat{S}_X(L_1+L_2) = \frac{n-(d_1+d_2)}{n} = \hat{S}_X(L_1) - \frac{d_2}{n}$$

$$\hat{S}_X\left(\sum_{j=1}^{i} L_j\right) = \frac{n-(d_1+d_2+\cdots+d_i)}{n} = \hat{S}_X\left(\sum_{j=1}^{i-1} L_j\right) - \frac{d_i}{n}$$

$$\hat{S}_X\left(\sum_{j=1}^{K} L_j\right) = \frac{n-\sum_{j=1}^{K} d_j}{n} = \hat{S}_X\left(\sum_{j=1}^{K-1} L_j\right) - \frac{d_K}{n}$$

If $L_i \equiv L$, can find $\hat{S}_X(0)=1$, $\hat{S}_X(L)$, $\hat{S}_X(2L), \ldots, \hat{S}_X(KL)$.

Linear interpolation is used to find $\hat{S}_X(t)$ between endpoints, giving the ogive empirical survival function.

Let $t_L \le t < t_U$, then

$$\hat{S}_X(t) = \frac{(t_U - t)\,\hat{S}_X(t_L) + (t-t_L)\,\hat{S}(t_U)}{t_U - t_L}.$$

ex] Know for exam 2

100000 lives from exact age 50 are observed for 30 years

	age last birthday	deaths
10	50–59	1700
10	60–69	4650
5	70–74	5520
5	75–79	9680

$\underbrace{\qquad\qquad\qquad}_{\text{given}}$

$\hat{S}_{50}(0) = 1$

$\hat{S}_{50}(10) = 1 - \frac{1700}{100000} = 0.9830$

$\hat{S}_{50}(20) = .983 - \frac{4650}{100000} = .9365$

$\hat{S}_{50}(25) = .9365 - \frac{5520}{100000} = .8813$

$\hat{S}_{50}(30) = .8813 - \frac{9680}{100000} = .7845$

Note $\hat{S}_{50}(25) = \frac{100000 - (1700 + 4650 + 5520)}{100000}$

Go to $36\frac{1}{4}$

8] P662-3 The survival time of an individual is censored if the event of interest (death) has not been observed.

ex] 10000 65 year olds buy 10 year life insurance. During the 10 years, perhaps 20% die and 15% quit paying premiums. Then 80% of lifetimes are censored.

p 663 (36.9)

9] Let Y_i = time until event (death) for the ith person. Let Z_i = time ith person leaves "study" for any reason other than event of interest = time until person is censored. Then the (right censored) survival time

$T_i = \min(Y_i, Z_i)$,

Let $\delta_i = \begin{cases} 0 & \text{if } T_i \text{ is censored } (T_i = Z_i) \\ 1 & \text{if } T_i \text{ is not censored } (T_i = Y_i) \end{cases}$

10] convention order the survival times, if there are ties, put censored cases after uncensored cases. Denote a censored case by $T_i^* = T_i +$.

$T_1, T_2, T_3^*, \ldots, T_{n-1}^*, T_n$

11] Suppose $T \geq 0$, t = time

see ex above 8)

$30\frac{1}{4}$

ex continued Find $\hat{S}_{50}(t)$ at $t =$

a) 2, b) 17

soln $\hat{S}_{50}(t) = \dfrac{(t_U - t)\,\hat{S}_{50}(t_L) + (t - t_L)\,\hat{S}_{50}(t_U)}{t_U - t_L}$

a) $\hat{S}_{50}(\overset{t_L}{0}) = 1$ $\qquad \hat{S}_{50}(\overset{t_U}{10}) = .9830$

$$\hat{S}(\overset{t}{2}) = \frac{(10-2)\,1 + (2-0)\,.9830}{10-0}$$

$$= \frac{8 + 2(.983)}{10} = \boxed{.9966}$$

b) $\hat{S}_{50}(10) = .9830$ $\qquad \hat{S}_{50}(20) = .9365$

$$\hat{S}(17) = \frac{(20-17)\,.9830 + (17-10)(.9365)}{20-10}$$

$$= \frac{3(.9830) + 7(.9365)}{10} = \boxed{.9505}$$

$$[0,\infty) = \underset{\substack{\| \\ 0}}{\overset{I_1}{[t_0}}, t_1) \cup \overset{I_2}{[t_1, t_2)} \cup \dots \cup \overset{I_m}{[t_{m-1}}, \underset{\substack{\| \\ \text{"}\infty\text{"}}}{t_m)}.$$

want to estimate $S(t) = P(Y > t)$ but there may be censoring.

Let Y_i = time until event and $T_i = \min(Y_i, Z_i)$.

Let $P_j = P$ (surviving through I_j | alive at start of I_j)

$$= P(Y > t_j \mid Y > t_{j-1}) = \frac{P(Y > t_j, Y > t_{j-1})}{S(Y > t_{j-1})}$$

$$= \frac{S(t_j)}{S(t_{j-1})}. \quad \text{so} \quad P_1 = \frac{S(t_1)}{S(t_0)} = S(t_1)$$

since $S(0) = S(t_0) = 1$. $\therefore$

$$S\,t_j = \underbrace{S(t_1)\,\frac{S(t_2)}{S(t_1)}\,\frac{S(t_3)}{S(t_2)} \cdots \frac{S\,t_{j-1}}{S(t_{j-2})}\,\frac{S(t_j)}{S(t_{j-1})}}_{\text{telescoping product}}$$

$$= p_1 p_2 \cdots p_j = \prod_{i=1}^{j} p_i .$$

Let $\hat{p}_i = 1 - \dfrac{\#\text{ dying in } I_i}{\#\text{ with potential to die in } I_i}$

[2] Let $(t_{(j)}, \gamma_j)$ correspond to ordered

$T_1, T_2, T_3^*, \ldots, T_{n-1}^*, T_n$

$t_{(j)}$	$t_{(1)}$	$t_{(2)}$	$t_{(3)}$		$t_{(n-1)}$	$t_{(n)}$
γ_j	1	1	0		0	1 .

[3] Let $t_1, \ldots, t_m$ be the ordered times where an event (death) occurred.

Let $r_i = \#$ at risk at t_i = # alive and uncensored just before t_i: $\quad r_i = \sum_{j=1}^{n} I(t_{(j)} \ge t_i)$.

Let $d_i = \#$ of events (deaths) at t_i.

Then $d_i \ge 1$ and if $d_i = 1$ for $i = 1, \ldots, m$, there are no ties. If $d_i > 1$ for some i, then there are ties.