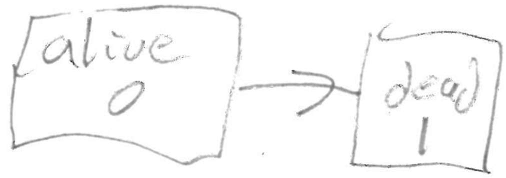
# Ch 8 Multiple State Models

ex] alive–dead model




The states are 0 = alive and 1 = dead.

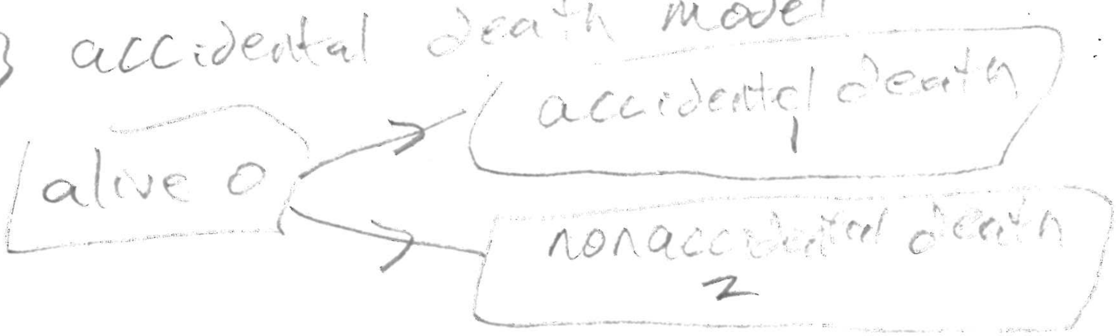
Let life $x \ge 0$ at time $t=0$.

Let $Y(t) = \begin{cases} 0 & \text{if } x \text{ is alive at } x+t \\ 1 & \text{dead} \end{cases}$

$\{Y(t)\}_{t\ge 0}$ is a continuous time stochastic process. Once $Y(t)=1$, say at $t_0$, then $Y(t)=1$ $\forall t \ge t_0$.

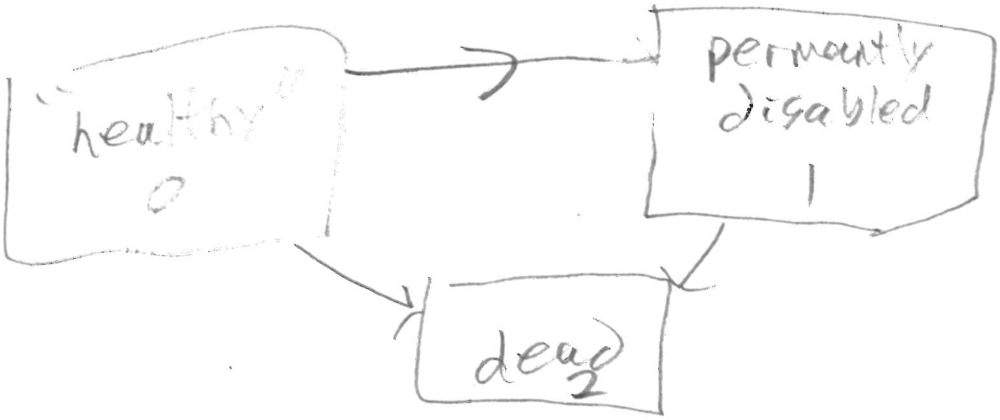
If we only look at $t=0,1,2,\dots 120-x$ years then $Y(t)$ is a lot like discrete Markov chain.

ex] accidental death model




$Y(t) = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$ if $x+t$ is in state $i$, $i=0,1,2$

ex) permanent disability model

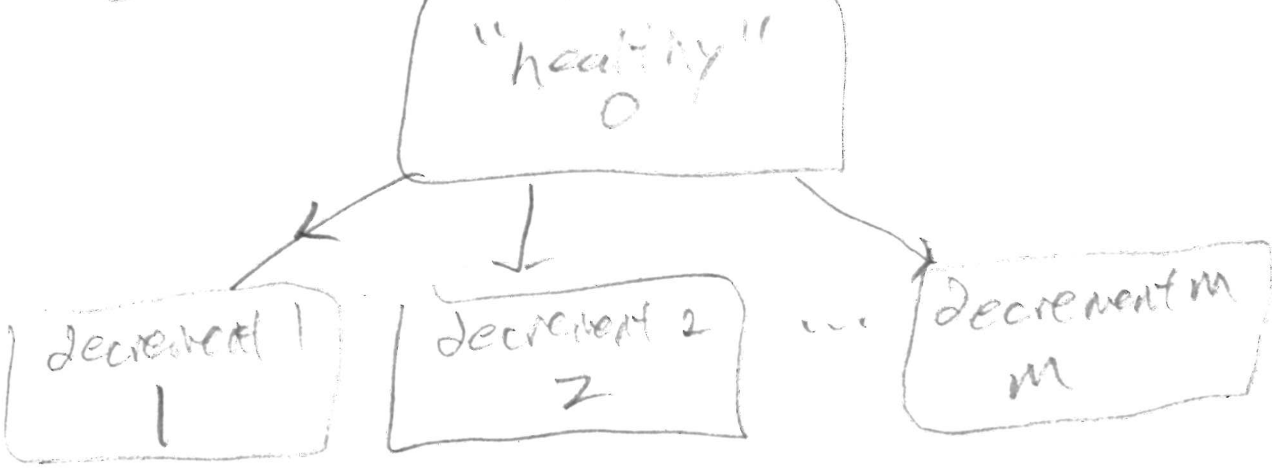



A permantly disabled individual can not become healthy.

ex) sickness – death


A sick person can get healthy. (can replace sick by disabled)

ex) multiple decrement model




eg $m=2$ 1 death by natural causes
2 death by accidental causes

1) Assume the multiple state model has $m+1$ states $0, 1, \dots, m$.

For each $t \geq 0$, $Y(t) \in \{0, 1, \dots, m\}$.

$\{Y(t)\}_{t \geq 0}$ is a continuous stochastic process.

Markov Property: For any states $i$ and $j$ and times $t, s \geq 0$,

$P(Y(t+s) = j \mid Y(t) = i)$ is well defined and does not depend on any information about the process before time $t$ (Given $Y(t)$, $Y(t+s) \perp\!\!\!\perp Y(x)$ for $x < t$).

Hence $\{Y(t)\}_{t \geq 0}$ is a Markov process.

For any positive interval of length $h$

$P$[2 or more transitions within a time period $h$]

$= o(h)$ where a function $g(h) = o(h)$ (little o h)

if $\lim_{h \to 0} \frac{g(h)}{h} = 0$.

Intuition $g(h) = o(h)$ if $g(h) \to 0$ faster than $h \to 0$

If $h$ is tiny. $P\{2$ or more transitions in time $t$ to $t+h\} = o(h)$ which is "vanishing small" as $h \to 0$.

ex} Let $k$ be a constant

$g(h) = h^k = o(h)$ iff $k > 1$.

2}* For states $i, j$ in a multiple state model, for $x, t > 0$ define

a) ${}_tp_x^{ij} = P[Y(x+t) = j \mid Y(x) = i]$

b) ${}_tp_x^{\overline{ii}} = P[Y(x+s) = i$ for all $s \in [0,t] \mid Y(x) = i]$

So ${}_tp_x^{ij}$ = prob life currently aged $x$ and currently in state $i$ is in state $j$ at age $x+t$.

Here $j \in \{0, 1, \ldots, m\}$ so ${}_tp_x^{ii}$ has $j = i$.

${}_tp_x^{\overline{ii}}$ is the prob that a life currently aged $x$ and currently in state $i$ stays in state $i$ continuously from age $x$ to age $x+t$.

c) $\mu_x^{ij} = \lim\limits_{h \to 0^+} \dfrac{{}_hp_x^{ij}}{h}$ for $i \neq j$.

3) For all states $i$ and $j$ and all ages $x \geq 0$, assume that ${}_tp_x^{ij}$ is a differentiable function of $t$.

This assumption is sufficient for $\mu_x^{ij}$ to exist.

ex) $\boxed{\begin{matrix}\text{alive}\\ 0\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{dead}\\ 1\end{matrix}}$

${}_tp_x^{00} = {}_tp_x$, ${}_tp_x^{01} = {}_tq_x$

${}_tp_x^{10} = 0$ (can't go from dead to alive)

$\mu_x^{01} = \mu_x$.

4) $\mu_x^{ij}$ = <u>force of transition</u> (or transition intensity) between states $i$ and $j$ at age $x$.

5) An <u>absorbing state</u> is a state from which no exit is possible. If state $i$ is an absorbing state, then $\mu_x^{ij} = {}_tp_x^{ij} = 0 \quad \forall j \neq i$.

$P(Y(x+t) = j \mid Y(x) = i) = 0 \quad \forall j \neq i$

$P(Y(x+t) = i \mid Y(x) = i) = 1$.

ex) death is an absorbing state

ex) decrement $j$ = failure of status due to cause $j$ is an absorbing state, $j = 1, \ldots, m$

6) Suppose $\{Y(t), t \geq 0\}$ takes on values in the nonnegative integers. Then $\{Y(t), t \geq 0\}$ is a continuous-time Markov Chain if for all $s, t \geq 0$ and nonnegative integers $i, j, y(u)$, $0 \leq u < t$,

$$P[Y(t+s) = j \mid \overbrace{Y(t) = i}^{\text{present}}, \underbrace{Y(u) = y(u), 0 \leq u < t}_{\text{past}}]$$

$$= P[Y(t+s) = j \mid Y(t) = i].$$

Taking $P(Y(t) = k) = 0$ for $k > m$, the multiple state model with states $0, 1, \ldots, m$ is a continuous Markov Chain.

The Markovian property is that the conditional distribution of the future $Y(t+s)$ given the present $Y(t)$ and the

the past $Y(u)$, $0 \le u < t$,

depends only on the present and is independent of the past.

7) multiple decrement model  m decrements

$${}_tp_x^{(\tau)} = {}_tp_x^{00} \qquad (\tau) \to 00$$

$$\mu_{x+t}^{(j)} = \mu_{x+t}^{0j} \qquad \underbrace{(j)}_{\text{what I used}} \to \underbrace{0j}_{\text{text notation}}$$

8) $${}_tp_x^{\overline{ii}} = \exp\left[ -\int_0^t \sum_{\substack{j=0 \\ j \ne i}}^{m} \mu_{x+s}^{ij} \, ds \right]$$

9) <u>Kolmogorov's forward equations</u>

$$\frac{d}{dt} {}_tp_x^{ij} = \sum_{\substack{k=0 \\ k \ne j}}^{m} \left( {}_tp_x^{ik} \mu_{x+t}^{kj} - {}_tp_x^{ij} \mu_{x+t}^{jk} \right)$$

M 402

25) 582 23$\frac{1}{2}$

52)

# Section 8.10 Markov Chains

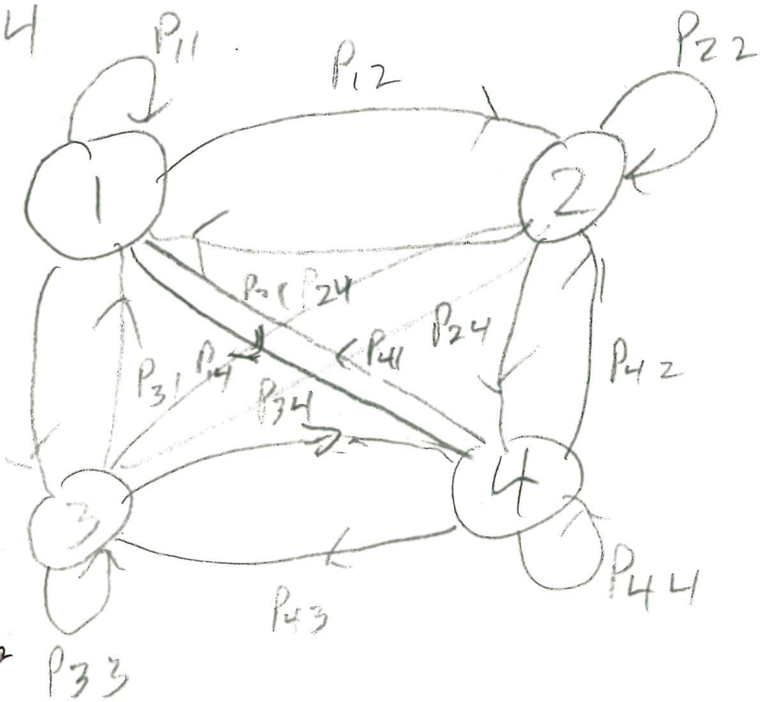
~~Appendix A~~ 1] ~~p565~~ P33 A (finite) Markov Chain $\{X_n : n = 0,1,2,\dots\}$ is a discrete stochastic process for which time only takes on integer values. $X_n$ will have J possible values $1,\dots,J$ where often $J = 2, 3$ or $4$. If $X_n = i \in \{1,\dots,J\}$, then the Markov chain is in state i at time n. For a Markov chain, the $P(X_{n+1} = j \mid \text{past})$ depends only on the state the process was in at time $t = n$, not on $t < n$.

so $P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \dots, X_1 = i_1, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i)$ Markov Property for any $n \geq 1$.

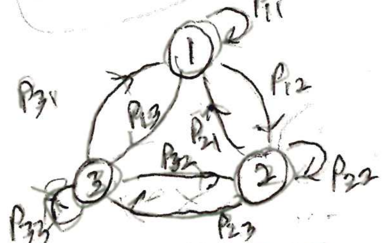
2] ~~P566~~ P35 The transition probability $P_{ij} = P(X_{n+1} = j \mid X_n = i)$

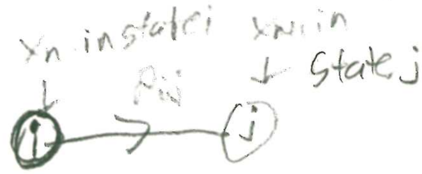
The transition probability matrix $\mathbb{P} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1J} \\ P_{21} & P_{22} & \cdots & P_{2J} \\ \vdots & & & \\ P_{J1} & P_{J2} & \cdots & P_{JJ} \end{bmatrix}$.

ex) M402 $J = 4$




use J=3
less
clutter




⇐ Transition diagram, book ex on p 569 leaves off $P_{ii}$ which can be found using 3} below. It leaves off an arrow for $P_{ij}$ if $P_{ij} = 0$, $i \neq j$.

~~p566~~ p35

3} The sum of the prob's in any row of $\mathbb{P}$ is

$\sum_{j=1}^{J} P_{ij} = 1$ for row $i = 1, \ldots, J$. Given

$X_n = i$, then the $X_{n+1}$ state will be $j \in \{1, \ldots, J\}$

with prob $= 1$.

4) p37 * $P^n_{ij} = \mathbb{P}(X_{m+n} = j \mid X_m = i) = {}_nP^{ij}$

$P^n_{ij}$ is the ij th entry of $\mathbb{P}^n = \underbrace{\mathbb{P} \cdots \mathbb{P}}_{\text{multiply n times}} = [P^n_{ij}]$

5] State j is accessible from state i

if $P^n_{ij} > 0$ for some $n \geq 0$. State j is accessible from state i iff, starting in i, it is possible that the process will ever enter state j in a finite # of steps.

6] Two states $i$ and $j$ that are accessible to each other communicate, written $i \leftrightarrow j$, (53.5)

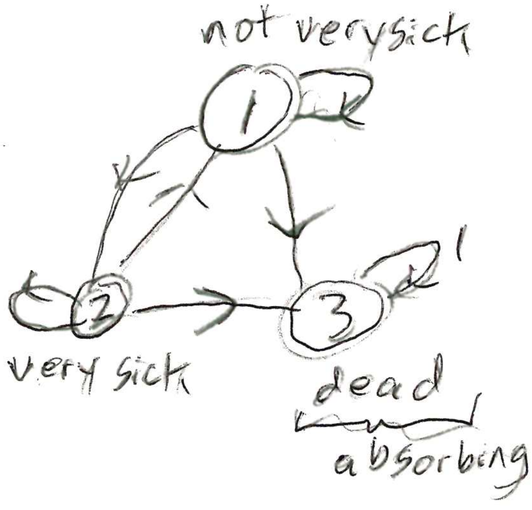
7] i) State $i$ communicates with state $i$, $i=1,\dots,J$.
ii) If state $i$ communicates with state $j$, then state $j$ i.
iii) If state $i$ j
and state $j$ $k$,
then state $i$ $k$.

8) States that communicate with each other form an equivalence class. A Markov chain is irreducible if there is only one class, so all states communicate with each other.

9] For state $i$, let $r_i$ denote the prob, starting in state $i$, the process will ever reenter state $i$. State $i$ is recurrent if $r_i = 1$ and transient if $r_i < 1$. State $i$ is absorbing if $P_{ii} = 1$. Once in an absorbing state, the Markov chain stays in the state for all subsequent time periods.

An absorbing state is recurrent with $r_i = 1$, but not all recurrent states are absorbing.

ex]




10] In $\mathbb{P}$, an absorbing state has $p_{ii} = 1$ and all other entries $p_{ij} = 0$ in the ith row.

11] A recurrent state will be revisited infinitely often.
A transient state is not certain to be revisited and will be visited only a finite # of times.
Starting in transient state i, the # of time periods $N$ the process will be in state i, including the initial time, is geometric with finite mean $E(N) = \frac{1}{1-r_i}$.

12] For an irreducible Markov chain with J states, all states are recurrent.

13] If state i is recurrent and $i \leftrightarrow j$, then state j is recurrent
transient transient.

14] State i is recurrent if $E(N) = \infty$
transient $E(N) < \infty$.

ex) $\mathbb{P} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}$ (S4.5)

State 3 is absorbing, $1 \leftrightarrow 2$, $1 \not\leftrightarrow 3$, $2 \not\leftrightarrow 3$.
$\{1,2\}$, $\{3\}$ are the 2 classes so
$\mathbb{P}$ is not irreducible

ex) $\mathbb{P} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.001 & 0.999 \end{bmatrix}$ is irreducible with class $\{1,2,3\}$

15) Every Markov chain must have at least one recurrent state in order for the process to continue for an indefinite length of time.

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16) *Let $\underset{\sim}{\Pi}_n = (\Pi_{1n}, \cdots, \Pi_{Jn})$ represent the probabilities of being in states 1 to J at time n.
Let $\underline{\Pi_0} = (\Pi_{10} \cdots \Pi_{J0})$ where $\Pi_{i0} = P(X_0 = i)$ = prob process in state i at the start. Ex) $\underline{\Pi_0} = (0, \cdots, 0, 1, 0, \cdots, 0)$ if the process is in state j at time $n=0$.
$\hphantom{(0, \cdots, 0,} \uparrow_j$

Then $\underline{\Pi_n}$ is the <u>state vector</u> at time n

and $\underline{\Pi_n} = \underset{\sim}{\Pi_0} \underline{\mathbb{P}^n = \underline{\Pi}_1 \mathbb{P}^{n-1} = \underset{\sim}{\Pi}_2 \mathbb{P}^{n-2} = \underline{\Pi}_K \mathbb{P}^{n-K} = \underline{\Pi}_{n-1} \mathbb{P}}$

$\underline{\Pi}_{n+1} = \underline{\Pi}_n \mathbb{P}$

$\underline{\Pi_0}$ is the <u>initial distribution</u> of the Markov Chain