M402 28

17) * p38 For a nonhomogeneous Markov chain, the matrix of transition probabilities $\mathbb{P}^{(k)} = \mathbb{P}_k$ depends on the kth step of the process. Then

$\underset{\sim}{\Pi}_n$ = state vector at time n satisfies

$$\underset{\sim}{\Pi}_n = \underset{\sim}{\pi}_0 \underbrace{\mathbb{P}^{(1)} \mathbb{P}^{(2)} \cdots \mathbb{P}^{(n)}}_{\text{these need to be given}}$$

text uses $\mathbb{P}^{(0)} \cdots \mathbb{P}^{(n-1)}$

Note $\mathbb{P}^n \neq \mathbb{P}^{(n)}$

M402

18) * Typically the initial distribution $\underset{\sim}{\Pi}_0 = (\pi_{10}, \ldots, \pi_{J0})$ is given, <u>or</u> you are told the chain is in state j so $\underset{\sim}{\Pi}_0 = (0, \ldots, 0, \underset{\substack{\uparrow \\ j}}{1}, 0, \ldots, 0)$ if states are $1, \ldots, J$

ex) p38 $\mathbb{P}^{(1)} = \begin{bmatrix} .6 & .4 \\ .7 & .3 \end{bmatrix}$, $\mathbb{P}^{(2)} = \begin{bmatrix} .5 & .5 \\ .8 & .2 \end{bmatrix}$

If process begins in state 2, what is the prob the process will be in state 1 after 2 steps? Soln: $\underset{\sim}{\Pi}_0 = (0, 1)$

(55.3)

$\underset{\sim}{\pi}_2 = (\pi_{12}, \pi_{22})$. want π_{12}.

Now $\underset{\sim}{\pi}_2 = \underset{\sim}{\pi}_0 \, \mathbb{P}^{(1)} \mathbb{P}^{(2)} = [0 \quad 1] \begin{bmatrix} .6 & .4 \\ .7 & .3 \end{bmatrix} \mathbb{P}^{(2)}$.

$$= [.7 \quad .3] \begin{bmatrix} .5 & .5 \\ .8 & .2 \end{bmatrix} = (.35+.24 \quad .35+.06)$$

$$= (.59 \quad .41)$$

So $\boxed{\pi_{12} = .59}$

common error: give vector

ex] MLC 151 For a multistate model with 3 states Healthy (0), Disabled (1) and Dead (2), for $k = 0, 1$

k^{2013} 2010 → transition prob matrix is

$p^{00}_{x+k} = 0.7$

$p^{01}_{x+k} = 0.2$

$p^{10}_{x+k} = 0.1$

$p^{12}_{x+k} = .25$

or $\mathbb{P} =$
$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & .7 & .2 & .1 \\ 1 & .1 & .65 & .25 \\ 2 & 0 & 0 & 1 \end{array}$$

There are 100 lives at the start, all healthy. The future states are ind (for each life). Calculate the variance of the number of the original 100 lives who die within the first 2 years.

Soln] ways to go from (0) to (2) in two years

initial	Year1	Year2	prob	
0	→ 0	→ 2	(.7)(.1)	= 0.07
0	→ 1	→ 2	(.2)(.25)	= 0.05
0	→ 2		.1	= .1
				0.22

(0 → 2 → 2 → .1(1) = .1)

Let X = # who die in 2 years

X ~ binomial (n = 100, p = 0.22)

$V(X) = np(1-p) = 100(0.22)0.78 = \boxed{17.16}$

also find $\pi_2 = \pi_0 \mathbb{P}\mathbb{P}$ to get .22

mult choice had A) 11 B) 14 C) 17 D) 20 E) 23

so $\boxed{C}$

ex) MLC 180 A certain species of flower has 3 states: sustainable, endangered, and extinct. Transitions between states are modeled as a non homogeneous Markov chain with transition probability matrices

Start in endangered

56.5

$$Q_0 = \begin{array}{ccc} \text{Sustainable} & \text{endangered} & \text{Extinct} \\ \end{array}$$
$$Q_0 = \begin{bmatrix} 0.85 & 0.15 & 0 \\ 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}$$

For 1st transition, can only go to endangered or extinct.

$$Q_1 = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}$$

For 2nd transition if in endangered; can't get to extinct if state becomes sustainable but can also go to extinct or endangered

$$Q_2 = \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}$$

for 3rd transition if in endangered; can only get to extinct 1 way

$$Q_n = \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad n = 3, 4, 5, \ldots$$

after 3rd transition, can't go extinct if sustainable or endangered

Calculate the prob that a species endangered at time 0 will ever become extinct.

soln] The flower needs to go extinct in the 1st 3 transitions.

$$\underset{\sim}{\pi_3} = \underset{\sim}{\pi_0}\, Q_0 Q_1 Q_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.85 & .15 & 0 \\ 0 & 0.7 & .3 \\ 0 & 0 & 1 \end{bmatrix} Q_1 Q_2$$

$$= \begin{bmatrix} 0 & 0.7 & 0.3 \end{bmatrix} \begin{bmatrix} x & x & x \\ 0.1 & 0.7 & 0.2 \\ 0 & 0 & 1 \end{bmatrix} Q_2$$

$$= \begin{bmatrix} \underset{.7(.1)}{.07} & \underset{.7(.7)}{.49} & \underset{.7(.2)+.3}{.44} \end{bmatrix} \begin{bmatrix} .95 & .05 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} .07(.95)+.49(.2) & .07(.05)+.44(.7) & \underbrace{.49(.1)+.44}_{\text{only needed this}} \end{bmatrix} =$$

$$\begin{bmatrix} .1645 & .3465 & .4890 \end{bmatrix}$$

So $\boxed{.489}$

Or look at paths starting in End at $n=0$

Ext		=	0.3
End → Ext	.7 (.2)	=	0.14
End → End → Ext	.7(.7)(.1)	=	0.049
transition 1 2 3			0.489

begin exam 3 material

<u>Policy values</u> also called

Ch 7 <u>Contingent Contract Reserves</u> and Benefit Reserves

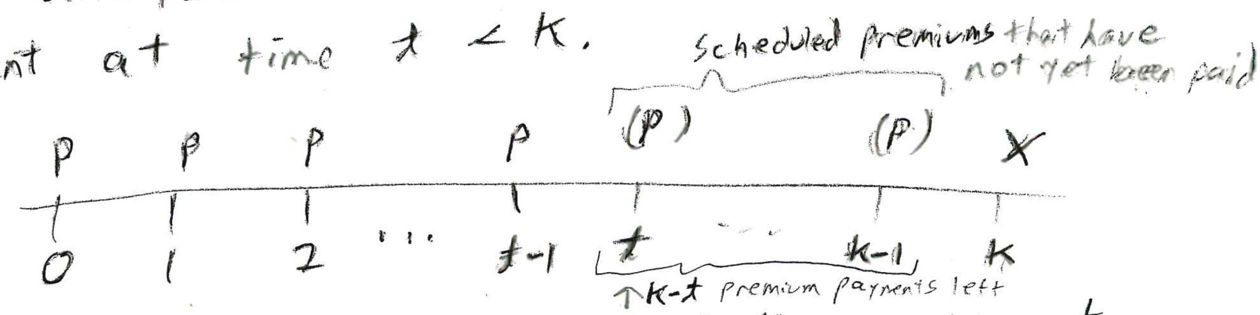
If the insurance benefit is B, multiply the unit benefit formula $_tV$ by B. (*)

(unit Premium is multiplied by B and APV of unit benefit is multiplied by B so $B\,_tV = B$ [unit APV − APV future unit Premiums])

C

~~Ch 8) Benefit Reserves~~

Idea: suppose premiums P are paid at time 0, 1, ..., k-1 and benefit X is paid at time k, want to analyze the status of the funding arrangement at time $t < k$.



From the equivalence Principle, $P\,\ddot{a}_{\overline{k|}} = X v^k$ at time 0.

Recall $(1+i)^{-t} = v^t$ and $(1+i)^t = v^{-t}$

So $P\,\underbrace{\ddot{a}_{\overline{k|}}(1+i)^t}_{\ddot{s}_{\overline{t|}} + \ddot{a}_{\overline{k-t|}}} = X v^{k-t}$ at time t.

(where $\ddot{s}_{\overline{t|}} = v^{-1} + v^{-2} + \dots + v^{-t}$, $\ddot{a}_{\overline{t|}} = 1 + v + v^2 + \dots + v^{t-1}$)

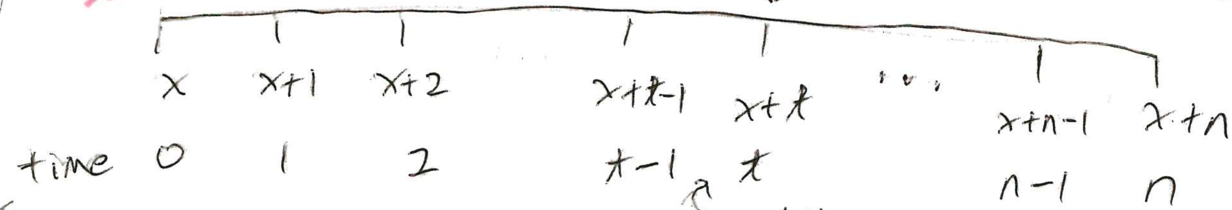
So $P\,\ddot{s}_{\overline{t|}} = X v^{k-t} - P\,\ddot{a}_{\overline{k-t|}}$ at time t.

LHS = financial status of funding scheme at the end of tth interval just before the payment for the $(t+1)$th interval is made

LHS $= {}_tV$ = tth terminal reserve (tth benefit reserve)

1] ${}_tV$ = (APV of future benefits − APV of future premiums) at time t. = policy value t years after policy was issued.

§10.1 ${}_tV$ is not a variance



(end of tth time period just before premium payment)

2] P235 Discrete insurance;
Let $t \geq 0$ be an integer, and insurance is paid at the end of the interval of failure at time K_x+1 where $K_x = \lfloor T_x \rfloor$

The NLP terminal reserve (Policy value) ${}_tV$ can be determined by the prospective method (only using future activity).

i) whole life $\quad {}_tV_x = A_{x+t} - P_x\, \ddot{a}_{x+t}$

Now let $t < n$. (If $t = n$, ${}_tV = 0$ or 1.)

ii) n year term $\quad {}_tV^1_{x:\overline{n}|} = A^1_{x+t:\overline{n-t}|} - P^1_{x:\overline{n}|}\, \ddot{a}_{x+t:\overline{n-t}|}$
$({}_nV^1_{x:\overline{n}|} = 0,\ t = n)$

iii) n year pure endowment $\quad {}_tV_{x:\overline{n}|}^{\;\;1} = A_{x+t:\overline{n-t}|}^{\;\;\;\;1} - P_{x:\overline{n}|}^{\;\;1}\, \ddot{a}_{x+t:\overline{n-t}|}$
$({}_nV_{x:\overline{n}|}^{\;\;1} = 1,\ t = n)$

iv) n year endowment $\quad {}_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\, \ddot{a}_{x+t:\overline{n-t}|}$
$({}_nV_{x:\overline{n}|} = 1,\ t = n)$

3] For h pay whole life, t an integer,

$$ {}^h_tV_x = \begin{cases} A_{x+t} - {}_hP_x\, \ddot{a}_{x+t:\overline{h-t}|} & \text{for } t < h \\ A_{x+t} & t \geq h \end{cases} $$

4] For n year deferred insurance, n premiums 59.9

$$ {}^{n}_{t}V(n|A_x) = \begin{cases} {}_{n-t|}A_{x+t} - {}_nP(n|A_x)\, \ddot{a}_{x+t:\overline{n-t}|}, & t<n \\ A_{x+t}, & t \geq n \end{cases} $$

Here t is an integer.

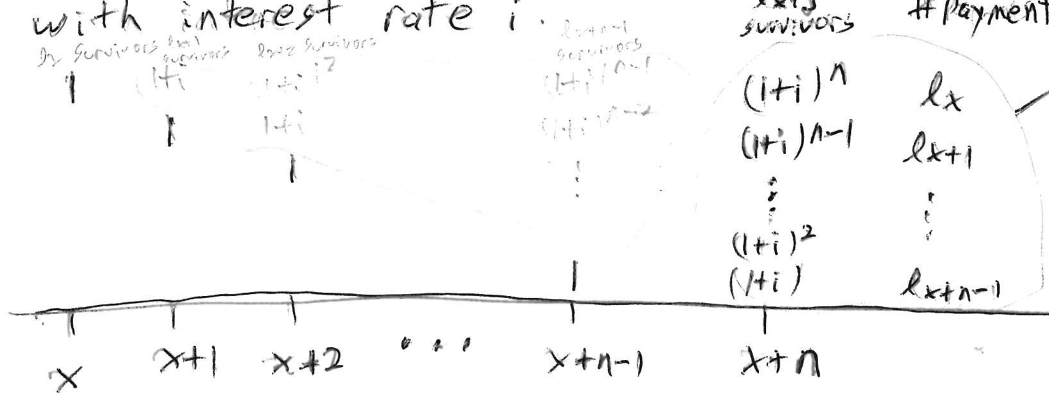
5) p237 The retrospective method for finding ${}_tV$ only examines the past $(x, x+t]$.

$$ {}_tV_x = P_x \ddot{s}_{x:\overline{t}|} - {}_tk_x = $$

$$ P_x \ddot{a}_{x:\overline{t}|} \frac{1}{{}_tE_x} - \overbrace{A^{1}_{x:\overline{t}|} \frac{1}{{}_tE_x}}^{} \quad \text{where } {}_tE_x = A_{x:\overline{t}|}^{\;\;1} $$

p123

6] p169 and problem 8.25 $\ddot{s}_{x:\overline{n}|}$ is the accumulated value (share) of a fund to be shared among l_{x+n} survivors if l_{x+j} survivors deposit 1 unit of money at time $x+j$, $j = 0, 1, \ldots, n-1$, with interest rate i.



accumulated fund to be shared is $\sum_{j=0}^{n-1} l_{x+j}(1+i)^{n-j} = W = l_{x+n} X$

$$ X = \ddot{s}_{x:\overline{n}|} = \frac{\sum_{j=0}^{n-1} l_{x+j}(1+i)^{n-j}}{l_{x+n}} = \frac{l_x(1+i)^n + l_{x+1}(1+i)^{n-1} + \cdots + l_{x+n-1}(1+i)}{l_{x+n}} $$

$$ = (1+i)^n \frac{l_x}{l_{x+n}} \frac{l_x(1+i)^n + \cdots + l_{x+n-1}(1+i)}{(1+i)^n l_x} = \frac{(1+i)^n l_x}{l_{x+n}} \frac{l_x + l_{x+1}v + \cdots + l_{x+n-1}v^{n-1}}{l_x} $$

$(1+i)^{-j} = v^j$

survival fn ${}_tP_x = \frac{l_{x+t}}{l_x}$

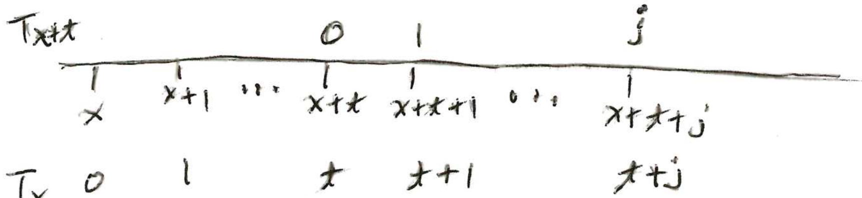
$$ = (1+i)^n \frac{l_x}{l_{x+n}} \left[1 + v P_x + \cdots + v^{n-1}\, {}_{n-1}P_x\right] = (1+i)^n \frac{l_x}{l_{x+n}} \ddot{a}_{x:\overline{n}|} = \frac{1}{{}_nE_x} \ddot{a}_{x:\overline{n}|} $$

402 32

7] $T_{x+t} = T_x - t$ ← given $X > x+t$.

$K_{x+t} = K_x - t$ for integer t

proof:



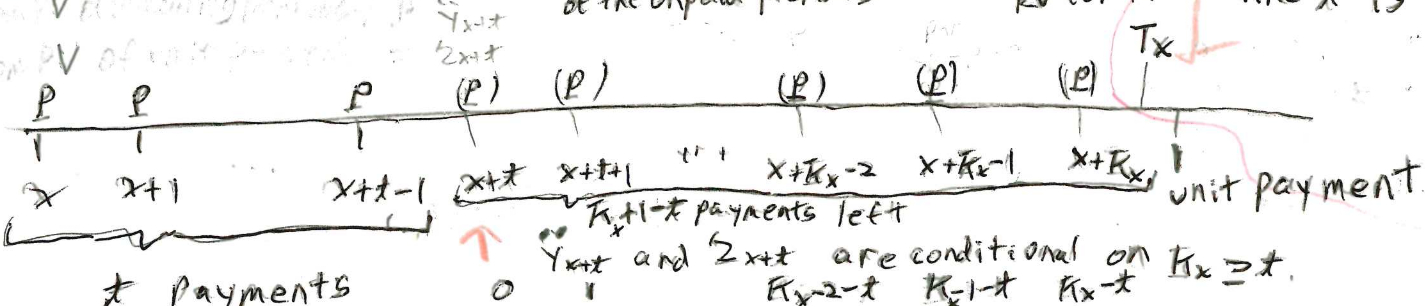
$$K_{x+t} = \lfloor T_{x+t} \rfloor \underset{\text{diagram}}{=} \lfloor T_x - t \rfloor \underset{\text{integer } t}{=} \lfloor T_x \rfloor - t = K_x - t \quad \text{for integer } t$$

8] Given $K_x \geq t$, the (random) value at time t of the $K_x + 1 - t$ premiums P yet to be paid at time K_x (random time of last premium payment) is $\underbrace{P\left[1 + v + v^2 + \dots + v^{K_x - t}\right]}$

$= P\, \ddot{a}_{\overline{K_x+1-t}|} = P\, \underbrace{\ddot{Y}_{x+t}}_{\text{RV for PV at time } t \text{ of the unpaid premiums}}$ $K_x - t + 1$ premiums left

RV for PV at time 0 is Z_x
RV for PV at time t is Z_{x+t}

At time t,



$\ddot{Y}_{x+t}$ and Z_{x+t} are conditional on $K_x \geq t$.

p241 whole life suppose $K_x \geq t$.

9] * Define ${}_tL_x = v^{K_x+1-t} - P_x\, \ddot{a}_{\overline{K_x+1-t}|} = Z_{x+t} - P_x\, \ddot{Y}_{x+t}$

= loss random variable at time t.

Then ${}_0L_x = L_x$ = loss variable from ch6 and

$$\underbrace{{}_tV_x}_{\text{terminal reserve = policy value}} = E({}_tL_x) = \underbrace{A_{x+t}}_{\text{APV of insurance at time } t} - \underbrace{P_x\, \ddot{a}_{x+t}}_{\text{APV of future payments (to be paid) at time } t} = \boxed{1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}}$$

$$= \frac{P_{x+t} - P_x}{P_{x+t} + d} = (P_{x+t} - P_x)\, \ddot{a}_{x+t} = \left(1 - \frac{P_x}{P_{x+t}}\right) A_{x+t} = 1 - (P_x + d)\, \ddot{a}_{x+t}$$

$$Var({}_tL_x) = V({}_tL_x) = \left(1 + \frac{P_x}{d}\right)^2 \left[{}^2A_{x+t} - (A_{x+t})^2\right] = \boxed{\frac{{}^2A_{x+t} - (A_{x+t})^2}{(1 - A_x)^2}}$$

$\ddot{Y}_{x+t} = \frac{1 - Z_{x+t}}{d}$

$$= \left(\frac{1}{d\, \ddot{a}_x}\right)^2 \left[{}^2A_{x+t} - (A_{x+t})^2\right] \text{ for unit payment}$$

10] P257 terminal policy values reserves for continuous payment insurance 60.5

with annual premiums put a bar over A

i) whole life $\quad {}_tV(\bar{A}_x) = \bar{A}_{x+t} - P(\bar{A}_x)\, \ddot{a}_{x+t}$

ii) n year term $\quad {}_tV(\bar{A}^{1}_{x:\overline{n}|}) = \bar{A}^{1}_{x+t:\overline{n-t}|} - P(\bar{A}^{1}_{x:\overline{n}|})\, \ddot{a}_{x+t:\overline{n-t}|}$

iii) n year endowment $\quad {}_tV(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+t:\overline{n-t}|} - P(\bar{A}_{x:\overline{n}|})\, \ddot{a}_{x+t:\overline{n-t}|}$

11] h pay n year term (continuous payment, h annual premiums)

$$
{}^{h}_{t}V(\bar{A}^{1}_{x:\overline{n}|}) = \begin{cases} \bar{A}^{1}_{x+t:\overline{n-t}|} - {}_hP(\bar{A}^{1}_{x:\overline{n}|})\, \ddot{a}_{x+t:\overline{h-t}|} & \text{for } t < h < n \\ \bar{A}^{1}_{x+t:\overline{n-t}|}, & h < t < n \end{cases}
$$

12] P258 n year deferred annuity (discrete) has n premiums over deferred period

$${}_tV({}_{n|}\ddot{a}_x) = {}_{n-t|}\ddot{a}_{x+t} - P({}_{n|}\ddot{a}_x)\, \ddot{a}_{x+t:\overline{n-t}|}$$

§10.3 13] continuous payment continuous funding

for discrete insurance puts ‾ over V, P and a.

So $\; {}_t\bar{V}_x = A_{x+t} - \bar{P}_x\, \bar{a}_{x+t}$

p249, 251   Fully continuous whole life insurance   402   B 5 6p

14] * continuous funding continuous payment, continuous whole life insurance. Given $T_x > t$,

$${}_t\bar{L}(\bar{A}_x) = v^{T_x - t} - [\bar{P}(\bar{A}_x)]\,\bar{a}_{\overline{T_x - t}|} = \bar{Z}_{x+t} - [\bar{P}(\bar{A}_x)]\,\bar{Y}_{x+t}$$

$${}_t\bar{V}(\bar{A}_x) = E[{}_t\bar{L}(\bar{A}_x)] = \bar{A}_{x+t} - [\bar{P}(\bar{A}_x)]\,\bar{a}_{x+t} = \boxed{1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}}$$

= APV of insurance at time t − APV of future premiums at time t payable at rate $\bar{P}(\bar{A}_x)$.

$$\mathrm{Var}[{}_t\bar{L}(\bar{A}_x)] = \left(\frac{1}{\delta\bar{a}_x}\right)^2 \left[{}^2\bar{A}_{x+t} - (\bar{A}_{x+t})^2\right] =$$

$$\boxed{\frac{{}^2\bar{A}_{x+t} - (\bar{A}_{x+t})^2}{(1 - \bar{A}_x)^2}}.$$

$$\underbrace{\bar{A}^{\,1}_{x:\overline{t}|}\left(\frac{1}{{}_tE_x}\right)}_{{}_t\bar{k}_x}$$

15] * Also ${}_t\bar{V}(\bar{A}_x) = [\bar{P}(\bar{A}_x)][\bar{s}_{x:\overline{t}|}] - {}_t\bar{k}_x$

$$= 1 - [\bar{P}(\bar{A}_x) + \delta]\,\bar{a}_{x+t} \overset{14]}{=} \boxed{1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}}$$

$$= [\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)]\,\bar{a}_{x+t}$$

Fully continuous (funding, payment, insurance are all continuous)

16) i) n year term $_t\bar{V}\left(\bar{A}^{\,1}_{x:\overline{n}|}\right) = \bar{A}^{\,1}_{x+t:\overline{n-t}|} - \bar{P}\left(\bar{A}^{\,1}_{x:\overline{n}|}\right)\bar{a}_{x+t:\overline{n-t}|}$ (6.5)

ii) n year pure endowment $_t\bar{V}_{x:\overline{n}|}^{\;\;\,1} = A_{x+t:\overline{n-t}|}^{\;\;\;\;\;\;\;\;\;\;\;1} - \bar{P}_{x:\overline{n}|}^{\;\;\,1}\,\bar{a}_{x+t:\overline{n-t}|}$

no bar

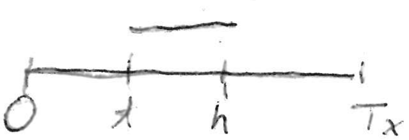
iii) n year endowment $_t\bar{V}\left(\bar{A}_{x:\overline{n}|}\right) = \bar{A}_{x+t:\overline{n-t}|} - \bar{P}\left(\bar{A}_{x:\overline{n}|}\right)\bar{a}_{x+t:\overline{n-t}|}$

iv) n year deferred annuity

$$_t\bar{V}\left(_{n|}\bar{a}_x\right) = {}_{n-t|}\bar{a}_{x+t} - \bar{P}\left(_{n|}\bar{a}_x\right)\bar{a}_{x+t:\overline{n-t}|}$$

v) h pay whole life insurance

$$^{h}_{t}\bar{V}\left(\bar{A}_x\right) = \begin{cases} \bar{A}_{x+t} - {}_h\bar{P}\left(\bar{A}_x\right)\bar{a}_{x+t:\overline{h-t}|} & t \le h \\ \bar{A}_{x+t} & t \ge h \end{cases}$$



future premiums from t to h

17) The fully continuous insurance and annuities tend to be the discrete formulas with bars over V, A, P, and a. The formulas tend to have the insurance or annuity in parentheses for the reserve $_tV$ and the premium P. For discrete insurance and annuities, tend to drop the parentheses but the subscripts are used for the reserve

${}_tV$ and premium P, An exception is discrete deferred insurance and annuities, which do use parentheses. Also pure endowment is both continuous and discrete but treated as discrete.

18) * For 4 insurances, $Var({}_tL) = \dfrac{{}^2A - (A)^2}{(1-A)^2}$ if the equivalence principle is used to determine premiums

i) Fully continuous whole life:

$$Var({}_tL) = Var\left[{}_t\bar{L}(\bar{A}_x)\right] = \frac{{}^2\bar{A}_{x+t} - \left[\bar{A}_{x+t}\right]^2}{(1-\bar{A}_x)^2}$$

$\boxed{\underline{\underline{E}}\ {}^2\bar{A}_x}$

← if $T_x \sim Exp(\mu)$

ii) Fully continuous n year endowment insurance $t < n$:

$$Var({}_tL) = var\left[{}_t\bar{L}(\bar{A}_{x:\overline{n}|})\right] = \frac{{}^2\bar{A}_{x+t:\overline{n-t}|} - \left[\bar{A}_{x+t:\overline{n-t}|}\right]^2}{(1-\bar{A}_{x:\overline{n}|})^2}$$

iii) Discrete whole life integral t

$$var({}_tL) = Var({}_tL_x) = \boxed{\frac{{}^2A_{x+t} - (A_{x+t})^2}{(1-A_x)^2}}$$

iv) Discrete n year endowment integral $t < n$:

$$Var({}_tL) = Var({}_tL_{x:\overline{n}|}) = \frac{{}^2A_{x+t:\overline{n-t}|} - (A_{x+t:\overline{n-t}|})^2}{(1-A_{x:\overline{n}|})^2}.$$

Take $t=0$ to get $V(0L) = V(L)$ from ch. 6. 62.5

At time t, let

19] p235 p254 ${}_tV$ = terminal reserve = policy value

A_{x+t} = APV of insurance (sometimes $\ddot{a}$ or $\bar{a}$)

$\ddot{a}$ or $\bar{a}_{x+t}$ a_{x+t} = APV of remaining unit Premiums

P_{x+t} = premium for $(x+t)$

i) The prospective formula is
$${}_tV = A_{x+t} - P_x\, a_{x+t}.$$

ii) Using $A_{x+t} = P_{x+t}\, a_{x+t}$,

$${}_tV = a_{x+t}\left(P_{x+t} - P_x\right)$$ is a premium difference formula.

iii) The paid up insurance formula is
$${}_tV = A_{x+t}\left(1 - \frac{P_x}{P_{x+t}}\right).$$

20] * For example,

i) discrete whole life ${}_tV_x = A_{x+t} - P_x\, \ddot{a}_{x+t} =$

$$\ddot{a}_{x+t}\left(P_{x+t} - P_x\right) = A_{x+t}\left(1 - \frac{P_x}{P_{x+t}}\right) = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$$