be chosen. Later on, one choice will be $Z = \frac{n}{n+\hat{k}}$.

Typically Z is close to 1 if n is large and the observed data $x_1, \ldots, x_n$ did not vary much from one period to the next. Z tends to be closer to 0 if n is small and $V(x_i)$ is large.

12] M is an estimate or a prior hypothesis of the premium (rate) to be charged in the absence of recent experience (data).

ex] A large pop of drivers has 0.2 accidents per driver per year (1 accident every 5 years). A randomly selected driver had 3 accidents in the past 5 years or 0.6 per year. Using

$Z(0.6) + (1-Z)\,0.2$ is likely a better estimate of this driver's future accident rate than 0.2 or 0.6.

13] (classical) limited fluctuation credibility attempts to limit the effect that random fluctuations (changes) in the observations will have on the estimates.
(See 20] for a formula for fluctuation.)

14] One question is how large should n be to assign full credibility $Z=1$. Want to determine the criterion for full credibility when estimating frequencies, severities or pure premiums (loss costs). Also want to determine Z if $Z \neq 1$: there is less data than needed for full credibility.

15] Let the aggregate claim for a policy be $S = \sum_{i=1}^{N} X_i$. Need $E(N)$, $V(N)$

$E(X_i)$, $V(X_i)$, $E(S) = E(N)\,E(X)$,

$V(S) = E(N)\,V(X) + [E(X)]^2\,V(N)$.

16] Credibility standards are established based on 2 parameters: a) the probability of being in a CI type interval (see 20)

z_p	1.645	1.96	2.576
p	90%	95%	99%

and b) the maximum amount of fluctuation to allow, eg $k = 0.05$.

17] Let e be the exposure needed for full credibility. Let W be a RV with $E(W)$, $V(W)$, $SD(W) = \sqrt{V(W)}$ and coefficient of variation $CV(W) = \dfrac{SD(W)}{E(W)}$.

Then $[CV(W)]^2 = \dfrac{V(W)}{[E(W)]^2}$. The <u>general formula for full credibility</u> is

$$e = n_0 \left[\frac{SD(W)}{E(W)}\right]^2 = n_0\,[CV(W)]^2$$

where $n_0 = \left(\dfrac{z_p}{k}\right)^2$. <u>Use for</u>

W	N	X	S
e	e_F	n_F	e_F

ex] Aggregate Claims $S \sim$ lognormal $(\mu = 7.5, \sigma = 2)$. Set the credibility standard so that actual aggregate claims are within 10% ($k=0.1$) of the expected aggregate claim 95% ($z_p = 1.96$) of the time. Determine the number of exposures needed for full credibility.

Soln: Let $e = e_F$. $E(S) = e^{\mu + 0.5\sigma^2}$

$$V(S) = e^{\sigma^2}(e^{\sigma^2}-1)e^{2\mu}$$

$$[CV(S)]^2 = \frac{V(S)}{[E(S)]^2} = \frac{e^{\sigma^2}(e^{\sigma^2}-1)e^{2\mu}}{e^{2\mu}e^{\sigma^2}} = e^{\sigma^2}-1$$

$= 53.5982$. So

$$e_F = \left(\frac{z_p}{k}\right)^2 [CV(S)]^2 = \left(\frac{1.96}{0.1}\right)^2 53.5982 =$$

$\boxed{20590.2845}$. (# policyholders: each policyholder has an aggregate claim)

18] Formulas are derived for e when experience is expressed in i) exposure units

$e = e_F$ = (expected) number of (members) risks for full credibility

ii) $e = n_F$ = (expected) number of claims needed for full credibility and

iii) (expected) aggregate losses $e = a_F = n_F E(X)$

= (number of claims)(expected loss per claim)

19) See 120) and 121) of exam 3 review

for formulas for e_F, n_F and a_F when

$N \sim Pois(\lambda)$. Horiz: "You want w to be within k of expected ρ of the time." Vert: "How many exposures are needed for full credibility?"

ex] Let $N \sim Pois(\lambda)$.

a) Let $W = S = \sum_{i=1}^{N} X_i$. Then

$$V(S) = \lambda V(X) + [E(X)]^2 \lambda = \lambda [(E(X))^2 + V(X)]$$

and $E(S) = \lambda E(X)$. Hence

$$[CV(S)]^2 = \frac{V(S)}{[E(S)]^2} = \frac{\lambda [[E(X)]^2 + V(X)]}{\lambda^2 (E(X))^2}$$

$$= \frac{1}{\lambda}\left[1 + \frac{V(X)}{[E(X)]^2}\right] = \frac{1 + [CV(X)]^2}{\lambda}.$$

So $e_F = n_0 [CV(S)]^2 = \frac{n_0}{\lambda}\left(1 + [CV(X)]^2\right)$.

b) Let $W = N$. Then

$$e_F = \left(\frac{z_p}{k}\right)^2 [CV(N)]^2 = \left(\frac{z_p}{k}\right)^2 \frac{\lambda}{\lambda^2} = \frac{n_0}{\lambda}.$$

c) Let $W = X$. Then $e = n_F$.

So $n_F = n_0 \left[CV(X)\right]^2$ and

$$e_F = \frac{n_F}{\lambda} = \frac{n_0}{\lambda}\left[CV(X)\right]^2.$$

20] The "CI" is $e\,E(W) \pm z_p\sqrt{e\,V(W)}$.

Want fluctuation $\dfrac{z_p\sqrt{e\,V(W)}}{e\,E(W)} \le K$.

Changing the inequality to an equality and solving for e gives $e = n_0\left[CV(W)\right]^2$.

If $W = N$ or $W = S$, then $e = e_F$.

If $W = X$, then $e = n_F$. Note:

eW is alot like $\sum_{i=1}^{e} W_i$ which has mean $e\,E(W)$ and variance $e\,V(W)$.

21]

dist	$1 + \left[CV(X)\right]^2$
$G(\alpha,\theta)$	$1 + \frac{1}{\alpha}$
$LN(\mu,\sigma)$	e^{σ^2}
pareto (α,θ)	$\frac{2(\alpha-1)}{\alpha-2}$

$Exp(\theta) \sim G(1,\theta) \to 2$

[22] Suppose you have $e < e_F$, $n < n_F$ or $a < a_F$ exposure units. $P_c = M + Z(T - M)$

where T is a statistic like $T = \bar{X}$ or $T = \Sigma X_i$ = total loss. M is the prior or manual rate or pure premium for what T estimates.

expected number of units	Z
e	$Z = \sqrt{e/e_F}$
n	$Z = \sqrt{n/n_F}$
a	$Z = \sqrt{a/a_F}$.

ex] Claim counts follow a Poisson dist. The standard for full credibility is 19544 expected claims. We observe 6000 claims and a total loss of 15,600,000 for a group of insureds. If our prior estimate of the total loss is 16 500 000, determine the limited fluctuation credibility estimate of the total loss for the insureds.

soln] total loss has $P_c = M + Z(\Sigma X_i - M)$

$M = 16\,500\,000$, $Z = \sqrt{\frac{n}{n_F}} = \sqrt{\frac{6000}{19\,544}}$

$P_c = 16{,}500{,}000 + \sqrt{\frac{6000}{19544}}\,(15{,}600{,}000 - 16{,}500{,}000) =$

16,001,332.11

65.5

§ 16.4.4

23} The Buhlmann method is a linear approximation of the Bayesian credibility method.

Let $\Theta = \theta_i$ correspond to risk class i.

Let the hypothetical mean μ_i be the mean of class i. The model or process is $X|\Theta$ (the likelihood if $n=1$).

Let $\mu = E[\mu(\Theta)] = E(X) = E\left[E[X|\Theta]\right] = EHM$ = overall mean = expected value of the process mean,

$v = EPV = E[V(X|\Theta)] = E[v(\Theta)]$ be the expected value of the process variance, and

$a = VHM = V[E(X|\Theta)] = V(\mu|\Theta)$ = variance of the hypothetical mean. Here $\mu(\Theta) = E(X|\Theta)$ and $v(\Theta) = V(X|\Theta)$.

So $\mu(\theta) = E(X|\Theta=\theta)$ and $v(\theta) = V(X|\Theta=\theta)$.

Buhlmann's $k = \dfrac{EPV}{VHM} = \dfrac{v}{a}$ and

$$z = \frac{n}{n+k} = \frac{n(VHM)}{n(VHM) + EPV} = \frac{na}{na+v}.$$ Then

for Buhlmann credibility, the credibility premium or Buhlmann credibility estimate is

$$P_c = z\bar{X} + (1-z)E(X) = E(X) + z[\bar{X} - E(X)]$$

ex) See ex done after (§12.4 17])

Claim sizes $X \sim$ single parameter Pareto $(\alpha = 3, \Theta)$

$\Theta \sim U[1,4]$. An insured selected at random submits 4 claims of sizes 2, 3, 5 and 7. Find Buhlmann's Z.

needed for P_c but not for Z

Soln] The model or "process" or "hypothesis" except for n

$X|\Theta \sim$ single parameter Pareto $(\alpha = 3, \Theta)$.

$$E(X|\Theta) = \frac{\alpha\Theta}{\alpha - 1} = 1.5\,\Theta$$

$$V(X|\Theta) = \frac{\alpha\Theta^2}{\alpha - 2} - [1.5\Theta]^2 = 0.75\,\Theta^2$$

$$EPV = E[V(X|\Theta)] = 0.75\,E[\Theta^2] = 0.75\left[V(\Theta) + [E(\Theta)]^2\right]$$

$$= 0.75\left[\frac{3^2}{12} + \left(\frac{5}{2}\right)^2\right] = \underline{5.25}.$$

$$VHM = V[E(X|\Theta)] = (1.5)^2\,V(\Theta) = (1.5)^2\,\frac{3^2}{12} = \underline{1.6875}.$$

$$\text{so } k = \frac{EPV}{VHM} = \frac{5.25}{1.6875} = 3\tfrac{1}{9} = 3.1111$$

$$Z = \frac{4}{4 + 3\frac{1}{9}} = \frac{9}{16} = \boxed{0.5625} = \frac{n}{n+k}$$

24] Buhlmann's method never assigns (66.5)
full credibility $Z=1$ $(k>0)$.
Classical limited fluctuations credibility
ignores the variance between hypothetical
means.

25] text notation

i) μ = EHM = $E(X)$ = overall mean = expected value of hypothetical mean

ii) a = VHM = $V(E(X|\Theta))$ = variance of hypothetical mean

iii) v = EPV = $E[V(X|\Theta)]$ = expected value of process variance

iv) Buhlmann's $k = \frac{v}{a} = \frac{EPV}{VHM}$

v) Buhlmann's credibility factor $Z = \frac{n}{n+k}$.
(uniform exposures)

26] Nonparametric empirical Bayes
estimation for Buhlmann credibility has

$$\hat{\mu} = \widehat{E(X)} = \overline{X} = \frac{1}{r}\sum_{i=1}^{r} \overline{X_i} = \frac{1}{nr}\sum_{i=1}^{r}\sum_{j=1}^{n} X_{ij}$$

$$\hat{v} = \widehat{EPV} = \frac{1}{r}\sum_{i=1}^{r} \underbrace{\frac{1}{n-1}\sum_{j=1}^{n}(X_{ij}-\overline{X_i})^2}_{\hat{\sigma}_{v_i}^2}$$

$$= \frac{1}{r}\sum_{i=1}^{r} \hat{\sigma}_{v_i}^2$$

$$\hat{a} = \widehat{VHM} = \frac{1}{r-1}\sum_{i=1}^{r}(\overline{X_i} - \overline{X})^2 - \frac{\hat{v}}{n},$$

where r = # policy holders

n = number of years of loss data.

If $\hat{a} < 0$, set $\hat{a} = 0$ and $\hat{Z} = 0$. Otherwise,

$$\hat{k} = \frac{\hat{v}}{\hat{a}} \quad \text{and} \quad \hat{Z} = \frac{n}{n+\hat{k}},$$

and the Buhlmann premium for policy holder i is

$$P_{c_i} = \hat{Z}\,\overline{X_i} + (1-\hat{Z})\,\overline{X}$$

$$= \overline{X} + \hat{Z}(\overline{X_i} - \overline{X}).$$

27] Semiparametric empirical Bayes estimation for Buhlmann credibility assumes that the number of claims for

each policyholder has a conditional Poisson (λ) distr (λ is a RV).
Each member has $\tilde{n}=1$ year of exposure.
Then $\hat{\mu} = \bar{X} = \hat{v}$ and

$$\hat{a} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 - \hat{v}. \quad \hat{k} = \frac{\hat{v}}{\hat{a}}$$

$$\hat{z} = \frac{1}{1+\hat{k}} = \frac{\tilde{n}}{\tilde{n}+\hat{k}}.$$ If $\hat{a} < 0$, set $\hat{a} = 0$ and $\hat{z} = 0$.

$$P_c^i = P_{c_i} = \hat{z} X_i + (1-\hat{z})\bar{X} = \bar{X} + \hat{z}(X_i - \bar{X}).$$

28) Nonuniform exposures: nonparametric empirical Bayes

If we have n_i years of data for group i with m_{ij} exposures for group i in year j and $m_i = \sum_{j=1}^{n_i} m_{ij}$ (exposure years) for group i over all n_i years, $\hat{\mu} = \bar{X} = \dfrac{\sum_{i=1}^{r} \sum_{j=1}^{n_i} m_{ij} X_{ij}}{m}$

where $m = \sum_{i=1}^{r} m_i$.

$$\hat{v} = \frac{\sum_{i=1}^{r} \sum_{j=1}^{n_i} m_{ij} (X_{ij} - \bar{X}_i)^2}{\sum_{i=1}^{r} (n_i - 1)}, \quad \hat{a} = \frac{\sum_{i=1}^{r} m_i (\bar{X}_i - \bar{X})^2 - \hat{v}(r-1)}{m - \frac{1}{m}\sum_{i=1}^{r} m_i^2}$$

If $\hat{a} < 0$, set $\hat{a} = 0$.