Note: (26) was the uniform exposure case with $n_i \equiv n$ and $m_{ij} \equiv m$.

29] $$\hat{Z}_i = \frac{m_i \hat{a}}{m_i \hat{a} + \hat{v}}, \quad i = 1, \ldots, r.$$

30] Back to Bayesian credibility. Poisson–Gamma

Suppose $N|\lambda \sim \text{poisson}(\lambda)$ where $\lambda \sim G(\alpha, \theta)$.

Then $N \sim NB(r = \alpha, \beta = \theta)$.

with $\underbrace{k \text{ claims in } n \text{ exposures}}$,

the posterior $\lambda | (n,k) \sim G\left(\alpha' = \alpha + k,\ \theta' = \frac{\theta}{1+n\theta}\right)$

where $\frac{1}{\theta'} = \frac{1}{\theta} + n = \frac{1+n\theta}{\theta}$.

The predictive distribution $N | (n,k) \sim NB(r = \alpha', \beta = \theta')$.

The n exposures could be n years for one insured, or n insureds for one year, or the sum of n_i insureds for year i for years $1, \ldots, m$: $n = \sum_{i=1}^{m} n_i$.

The posterior mean $E[\lambda | \underline{x}] = \alpha' \theta'$

$$\underline{x} \leftarrow (n,k)$$

$$= P_c = \frac{(\alpha+k)\theta}{1+n\theta} = \frac{\alpha+k}{\frac{1}{\theta}+n} \underset{\bar{x} = \frac{k}{n}}{\overset{\uparrow}{=}} \frac{\alpha + n\bar{x}}{\frac{1}{\theta}+n} = \frac{\gamma}{\gamma+n}\frac{\alpha}{\gamma} + \frac{n}{n+\gamma}\bar{x}$$

where $\gamma = \frac{1}{\theta}$ and $Z = \frac{n}{n+\gamma}$.

Normal—Normal

31) If $X \sim N(\theta, \overset{v}{\sigma^2})$ and $\theta \sim N(\mu, \overset{a}{\tau^2})$, then $X \sim N(\mu, \sigma^2+\tau^2)$. The predictive distribution for $\theta \mid \underset{\sim}{X}=(X_1,\dots,X_n)$ is

$$N\left(\mu+Z(\bar{X}-\mu),\ (1-Z)\tau^2+\sigma^2\right)$$

with $Z = \dfrac{n}{n+\dfrac{\sigma^2}{\tau^2}}$. σ^2 is like v. τ^2 is like a.

Binomial—Beta

32} If $N \sim bin(q, m)$ and $q \sim beta(a,b)$ and there are k claims in the m exposures then $q \mid (m,k) \sim beta(a+k,\ b+m-k)$.

Here $q = P(\text{claim})$.

33} If the prior is $q \sim beta(a,b)$ and you observe n Bernoulli(q) (bin(q, m=1)) trials with k 1's (data $\underset{\sim}{x}$), then $q \mid \underset{\sim}{x} = q \mid (k,n) \sim beta(a+k,\ b+n-k)$. So the posterior mean $E[q \mid \underset{\sim}{x}] = \dfrac{a+k}{n+a+b}$.

Here $N|q \sim$ Bernoulli$(q) \sim$ bin$(q, m=1)$.

(The binomial case of 32) treats a $N|q \sim$ bin(q,m) as m Bernoullis.)

Then $N \sim$ Bernoulli$\left(\frac{a}{a+b}\right)$ and the predictive distribution $N|\underline{x} = N|(k,n) \sim$

Bernoulli$\left(\frac{a+k}{a+b+n}\right)$. (asymptotically Ber$(\bar{x})$)

34) Suppose $f(x)$ is given on its support $[0,1]$

$f(x)$	beta (a,b)
1	$a=b=1$ $U(0,1)$
$2x$	$a=2, b=1$
$2(1-x)$	$a=1, b=2$
$6x(1-x)$	$a=2, b=2$
$\frac{3}{2}\sqrt{x}$	$a=1.5, b=1$
$\frac{1}{2\sqrt{x}}$	$a=0.5, b=1$

Learn how to recognize the beta prior.

35) Back to limited fluctuations credibility.

The formula for full credibility is

$$e = n_0 \left(\frac{SD(W)}{E(W)}\right)^2 = n_0 [CV(W)]^2 .$$

Let $\mu_N = E(N)$

$\sigma_N = SD(N) = \sqrt{V(N)}$.

experience	number of claims $W=N$	claim size severity $W=X$	aggregate losses pure premium $W=S$
exposure units e_F	$n_0\left(\frac{\sigma_N^2}{\mu_N^2}\right)$	$\frac{n_0}{\mu_N}[CV(X)]^2$	$n_0\left(\frac{\sigma_N^2}{\mu_N^2} + \frac{[CV(X)]^2}{\mu_N}\right)$
number of claims n_F	$n_0\left(\frac{\sigma_N^2}{\mu_N}\right)$	$n_0[CV(X)]^2$	$n_0\left(\frac{\sigma_N^2}{\mu_N} + [CV(X)]^2\right)$
aggregate losses a_F	$n_0 E(X)\left(\frac{\sigma_N^2}{\mu_N}\right)$	$n_0 E(X)[CV(X)]^2$	$n_0 E(X)\left(\frac{\sigma_N^2}{\mu_N} + [CV(X)]^2\right)$

E3 rev (20)-(21) uses $N \sim Pois(\lambda)$

Note $n_F = \mu_N e_F = E(N) e_F$ and $a_F = E(X) n_F$.

If $W=N$, $e_F = n_0 [CV(N)]^2 = n_0 \left(\frac{\sigma_N^2}{\mu_N^2}\right)$.

If $W=X$, $n_F = n_0 [CV(X)]^2$.

If $W=S$, $e_F = n_0 [CV(S)]^2$ and $n_F = n_0 \mu_N [CV(S)]^2$

$$\boxed{\begin{aligned} E(S) &= E(N)E(X) \\ V(S) &= E(N)V(X) + [E(X)]^2 V(N) \end{aligned}} \rightarrow = \mu_N n_0 \frac{\mu_N V(X) + [E(X)]^2 \sigma_N^2}{\mu_N^2 [E(X)]^2}$$

$$= n_0 \left(\frac{V(X)}{[E(X)]^2} + \frac{\sigma_N^2}{\mu_N}\right) = n_0\left(\frac{\sigma_N^2}{\mu_N} + [CV(X)]^2\right)$$

36] Bernoulli short cut

Suppose $X \mid \Theta = \theta_i$ takes on 2 values

x	a	b
$P(X=x)$	p_a	p_b

Then $V(X \mid \Theta = \theta_i) = (b-a)^2 p_a p_b$ and the hypothetical mean

$$\mu_i = E(X \mid \Theta = \theta_i) = a p_a + b p_b .$$

ex] class A 90 risks

loss	prob
100	0.8
300	0.2

class B 10 risks ← $\pi(A) = .9, \pi(B) = .1$

loss	prob
100	0.7
400	0.3

Find Buhlmann's k for a randomly selected risk,

$$V(X \mid A) = (\overset{300-100}{200})^2 (.8).2 = 6400$$

$$V(X \mid B) = (300)^2 (.7).3 = 18900$$

} B. Shortcut

$$\mu_1 = E(X \mid A) = 100(.8) + 300(.2) = 140$$

$$\mu_2 = E(X \mid B) = 100(.7) + 400(.3) = 190$$

$$\text{So } v = EPV = E(V(X \mid \Theta)) = 6400(.9) + 18900(.1) = 7650$$

$$\text{and } a = VHM = V(E(X \mid \Theta)) = \overbrace{(190-140)^2 (.9).1}^{\text{B. shortcut}}$$

$$= 225$$

$$\text{So } k = \frac{v}{a} = \frac{7650}{225} = \boxed{34}$$

37]

	class 1	2	...	K
$E(X\mid\Theta=i)$ $=E(X\mid \text{class } i)$	μ_1	μ_2		μ_K
$V(X\mid \text{class } i)$	$V(X\mid 1)$	$V(X\mid 2)$		$V(X\mid K)$
prior $P(\text{class } i)$	π_1	π_2		π_K

$$\mu = E(X) = E[E(X\mid\Theta)] = \sum_{i=1}^{K}\mu_i\pi_i = E[\mu(\Theta)]$$

$$V = EPV = E[V(X\mid\Theta)] = \sum_{i=1}^{K} V(X\mid i)\,\pi_i = E[V(\Theta)]$$

$$a = VHM = V[\overbrace{E(X\mid\Theta)}^{w}] = V(W) = E(W^2)-(EW)^2 = E(W^2)-\mu^2$$

$$= \sum_{i=1}^{K}[\mu_i]^2\pi_i - \Big[\sum_i \mu_i\pi_i\Big]^2$$

last ex]

	A	B
μ_i	140	190
$V(X\mid i)$	6400	18900
$\pi_i = P(\text{class } i)$	0.9	0.1

often need to get from small table

$$V = EPV = \sum_i V(X\mid i)\pi_i = 6400(0.9) + 18900(0.1) = 7650$$

$$a = (140)^2(.9) + (190)^2(.1) - [140(.9) + 190(.1)]^2$$

$$= 21250 - [145]^2 = 225$$

Note that $\mu = \sum_i \mu_i\pi_i = 140(.9) + 190(.1) = 145.$

38} If Z is wanted

compute $Z = \dfrac{na}{na+v}$ rather than

computing K.

39] The exposure unit is the unit for which the credibility premium is changed.

ex) calculate number of claims per insured

exposure unit

ex) Calculate claim size per claim

ex) Number of claims for each member

often N

$X \sim Pois(\lambda)$, $\lambda \sim U(0, 0.1)$.

You are given 3 years experience for the group:

Year	# members	# claims
1	120	3
2	150	4
3	170	4

The group will have 200 members in year 4. Calculate the Buhlmann premium for the group in year 4.

Soln} $X|\lambda \sim Pois(\lambda)$ so

71.5

hypothetical mean $E(x|\lambda) = \lambda$

process variance $V(x|\lambda) = \lambda$.

$$\mu = E(x) = E(E(x|\lambda)) = E(\lambda) = 0.05$$

$$V = EPV = E[V(x|\lambda)] = E(\lambda) = 0.05$$

$$a = VHM = V[E(x|\lambda)] = V(\lambda) = \frac{(.1)^2}{12} = \frac{.01}{12}$$

so $K = \frac{V}{a} = \frac{.05}{.01/12} = 60$

The exposure unit is 1 member—year and the RV X is the number of claims per member—year.

So the number of exposures $n = 120+150+170$ $= 440$ and $Z = \frac{na}{na+V} = \frac{440}{440+60} = 0.88 = \frac{n}{n+k}$.

$$\left(\frac{na}{na+V}\,\frac{\frac{1}{a}}{\frac{1}{a}} = \frac{n}{n+k}\right)$$

The experience mean $\bar{X} = \frac{3+4+4}{440} = 0.025$.

So the premium for 1 member would be

$$E(x) + Z(\bar{x} - EX) = 0.05 + 0.88(0.025 - 0.05).$$

So the premium for the group is

$$\overbrace{200}^{\text{200 members}}\left[.05 + 0.88\,(0.025 - .05)\right] = 200(.028)$$

$$= \boxed{5.6}$$

40] Need to figure out the exposure unit and the RV X for which you are calculating the credibility unit.

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If the group has J members and

P_c' is the premium for 1 member,

then $J P_c'$ is the premium for the group.

41] Buhlmann credibility continuous prior:

The credibility premium is still the Buhlmann credibility estimate of the next claim count or claim size or aggregate loss.

You need to identify the hypothetical mean $(E(X|\Theta))$ and the process variance $(V(X|\Theta))$ and calculate $\mu = E(X) = E[E(X|\Theta)] = E[\mu(\Theta)]$

$v = EPV = E[V(X|\Theta)] = E[v(\Theta)]$ and

$a = VHM = V(E(X|\Theta))$.

Typically the prior is a brand name

continuous dist. 72.5

ex] Annual aggregate losses $\sim N(\mu, \sigma^2)$

$\mu \sim$ single parameter Pareto $(\alpha = 3, \theta = 100)$

$\sigma \sim G(\alpha = 10, \theta = 50)$ with μ independent

of σ, written $\mu \perp\!\!\!\perp \sigma$. Find Buhlmann's

K for annual aggregate losses.

soln] $X|(\mu, \sigma^2) \sim N(\mu, \sigma^2)$

The hypothetical mean $E[X|(\mu,\sigma^2)] = \mu$.

$$a = V[E(X|(\mu,\sigma^2))] = V(\mu) = \frac{\alpha\theta^2}{\alpha-2} - \left(\frac{\alpha\theta}{\alpha-1}\right)^2$$

$$= \frac{3(100)^2}{1} - \left(\frac{3(100)}{2}\right)^2 = 30000 - (150)^2$$

$$= 7500.$$

$$v = EPV = E[V(X|(\mu,\sigma^2))] = E(\sigma^2) = V(\sigma) + (E\sigma)^2$$

$$= \alpha\theta^2 + (\alpha\theta)^2 = 10(50)^2 + (500)^2 = 275000$$

$$\text{so } k = \frac{v}{a} = \frac{275000}{7500} = \boxed{36.6667}$$

42} Sometimes $S = \sum_{i=1}^{N} X_i$

where N and or X may have priors.

ex} $N \sim \text{geometric}(\beta = 0.1)$

$X \sim \text{pareto}(\alpha = 3, \Theta)$

$N \perp\!\!\!\perp X_i, \quad X_i \perp\!\!\!\perp X_j \quad i \neq j$

$\Theta \sim \text{weibull}(\tau = 0.25, \theta = 10)$.

calculate Buhlmann's k for the annual aggregate loss.

Soln $S|\Theta = \sum_{i=1}^{N} X_i$

The hypothetical mean $= E(S|\Theta)$

$$= E(N|\Theta)\, E(X|\Theta) = E(N)\, E[X|\Theta]$$

$$= \beta \frac{\Theta}{\alpha - 1} = 0.1 \frac{\Theta}{2} = 0.05\,\Theta.$$

$$a = V(E(S|\Theta)) = (0.05)^2\, V(\Theta) =$$

$$(.05)^2 (E\Theta^2 - [E\Theta]^2) = \ldots = \underline{9936}$$

$$V(S|\Theta) = E(N)\, V(X|\Theta) + V(N)\, [E(X|\Theta)]^2$$

$$= \beta \frac{\Theta^2 \alpha}{(\alpha-1)^2(\alpha-2)} + \beta(1+\beta) \left(\frac{\beta\,\Theta}{\alpha-1}\right)^2$$

$$= 0.1 \frac{3}{2^2(1)} \Theta^2 + 0.1(1.1)\left(\frac{0.1}{2}\right)^2 \Theta^2$$ 23.5

$$= 0.1025\, \Theta^2$$

so $V = E(V(X|\Theta)) = 0.1025\, E(\Theta^2)$

$$= 0.1025\, \theta^2\, \Gamma\left(1+\frac{2}{\tau}\right) = 0.1025\,(10)^2\, \Gamma\left(1+\frac{2}{0.25}\right)$$

$$= 10.25\, \Gamma(9) = 10.25\,(8!) = 413280$$

so $K = \frac{v}{a} = \frac{413280}{9936} = \boxed{41.5942}$

ex) Often the prior $\Theta \sim U(a,b)$:

$N \sim Pois(\Lambda)$, $\Lambda \sim U[0.1, 0.15]$.

Find Buhlmann's K for claim counts.

soln) $N|\Lambda \sim Pois(\Lambda)$

The hypothetical mean is $E(N|\Lambda) = \Lambda$.

The process variance $= V(N|\Lambda) = \Lambda$.

$$a = V(E(N|\Lambda)) = V(\Lambda) = \frac{(.05)^2}{12}$$

$$v = E(V(N|\Lambda)) = E(\Lambda) = \frac{.25}{2}$$

$$K = \frac{v}{a} = \frac{.25}{2}\,\frac{12}{(.05)^2} = \boxed{600}$$

43} p588 The Buhlmann credibility model

has 1 exposure in every time period.

The Buhlmann straub credibility model has m_j exposures in period j.

Assume $X_1, \dots, X_n$ are independent conditional on Θ. Let

$\mu(\theta) = E(X_j \mid \Theta = \theta)$ and

$$\frac{v(\theta)}{m_j} = V(X_j \mid \Theta = \theta).$$

The $X_j = \overline{W_j} = \frac{1}{m_j} \sum_{i=1}^{m_j} W_{ij}$ where conditional on Θ, the W_{ij} are independent with mean $\mu(\theta)$ and variance $v(\theta)$.

Then $\mu = E(\mu(\Theta)) = E(E(X_j \mid \Theta)) = E(X_j)$

$v = E(v(\Theta)) = E[V(X_j \mid \Theta)]$

$a = V(\mu(\Theta)) = V[E(X_j \mid \Theta)]$.

$Cov(X_i, X_j) \underset{\substack{\uparrow \\ i \neq j}}{=} a$, $V(X_j) = \frac{v}{m_j} + a$.

$K = \frac{v}{a}$, $z = \frac{m}{m+k}$, $m = \sum_{i=1}^{n} m_i$

$P_c^1 = z\bar{x} + (1-z)\mu = \mu + z(\bar{x}-\mu) =$
premium for 1 member of the group.

$\bar{x} = \sum_{j=1}^{n} \frac{m_j}{m} x_j$.

44] $x_j = \frac{1}{m_j}\sum_{i=1}^{m_j} w_{ij} = \bar{w}_j$ so $m_j x_j =$

total loss for group in year j.

45] The credibility premium charged to the group in year n+1 is

$$P_c = m_{\underbrace{n+1}_{\text{number of group members in year } n+1}} P_c^1 .$$

Skip Hewitt's model

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ex] Individual losses follow a pareto($\alpha=5$, Θ) dist with $\Theta \sim U[5000,7000]$.
For a group policy holder you observe the following experience

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	# individuals	# losses	total losses
year 1	200	10	8000
year 2	240	15	24000

Calculate the Buhlmann-Straub credibility estimate of the size of one loss from this group.

Soln. $\bar{X} = \dfrac{8000 + 24000}{10 + 15} = 1280$

Treat each loss as an obs so

$n = m = 10 + 15 = 25$,

$X_i \mid \Theta \sim \text{Pareto}(\alpha = 5, \Theta)$, $\Theta \sim U(5000, 7000)$

$$\mu(\Theta) = E(X_i \mid \Theta) = \frac{\Theta}{\alpha - 1} = \frac{\Theta}{4}$$

$$v(\Theta) = V(X_i \mid \Theta) = \frac{\Theta^2 \alpha}{(\alpha-1)^2(\alpha-2)} = \frac{5}{16(3)} \Theta^2 = \frac{5}{48} \Theta^2.$$

$$\mu = E[\mu(\Theta)] = E\left(\frac{\Theta}{4}\right) = \frac{5000 + 7000}{4(2)} = \frac{12000}{8} = 1500$$

$$v = E[v(\Theta)] = \frac{5}{48} E[\Theta^2] = \frac{5}{48}\left[V(\Theta) + (E\,\Theta)^2\right]$$

$$= \frac{5}{48}\left[\frac{(2000)^2}{12} + (6000)^2\right] = 3{,}784{,}722.222$$

$$a = V[\mu(\Theta)] = V\left(\frac{\Theta}{4}\right) = \frac{1}{16} V(\Theta) = \frac{1}{16} \frac{(2000)^2}{12} = 20833.3333$$

$K = \frac{v}{a} = \frac{3784,222.222}{20833.3333} = 181.643$ 75.5

$Z = \frac{n}{n+K} = \frac{25}{25+181.643} = 0.1210.$

$P_C' = \mu + z(\bar{x} - \mu) = 1500 + 0.121(1280 - 1500)$

$= \boxed{1473.384}$

46] §20.3.7 when the model distribution $X|\Theta$ is a linear exponential family and Θ is the conjugate prior, if the prior mean exists, then the Buhlmann credibility estimate = posterior mean = Bayesian premium.

See exam 3, review 142)

Ch21 Simulation

1] Idea: generate pseudo random (numbers) variables $X_1, \ldots, X_n$ that act like a sample $X_1, \ldots, X_n$ iid from a brandname distribution.