To get 38), sub in $X_{new} = (1+r)X$ in 37) and solve for X for getting the values of X for which the formulas apply. Then multiply the formulas by α.

39} *Under 38} for the per loss RV

$$E(Y^L) = \alpha(1+r)\left[E\left(X \wedge \frac{u}{1+r}\right) - E\left(X \wedge \frac{d}{1+r}\right)\right]$$

and for the per payment RV,

$$E(Y^P) = \frac{E(Y^L)}{1 - F_X\left(\frac{d}{1+r}\right)} = \frac{E(Y^L)}{S_X\left(\frac{d}{1+r}\right)}$$

Note: that $\alpha = 1$ for no coinsurance and $r = 0$ for no inflation. Take $u = \infty$ if there is no policy limit, so $E(X \wedge \infty) = E(X)$

40} *So if there is a deductible and a policy limit $u-d$ (with $u = u-d+d$), then

$E(Y^L) = E(X \wedge u) - E[X \wedge d]$ with

$E(Y^P) = \frac{E(Y^L)}{S_X(d)}$.

ex) with deductible d and policy limit $u-d$ M404 87

$u-d$ = maximum payment u = maximum covered loss $= u-d+d$

math403 E1d15 3) $X \sim Exp$ with mean 1000.

An insurance policy on X has deductible 500 and maximum policy payment per loss of 1500. Find Y^L and Y^P.

Soln $\theta = 1000$, $u-d = 1500$, $u = 1500+d = 2000$

$$E(Y^L) = E(X \wedge u) - E[X \wedge d]$$

$$= E(X \wedge 2000) - E[X \wedge 500] \quad \text{and}$$

$$E(X \wedge w) = \theta(1 - e^{-w/\theta}).$$

$$\text{so } E(Y^L) = 1000(1-e^{-2}) - 1000(1-e^{-\frac{1}{2}})$$

$$= 1000(e^{-\frac{1}{2}} - e^{-2}) = \boxed{471.1954}$$

$$S_X(d) = 1 - F_X(d) = e^{-d/\theta}$$

$$\text{So } E(Y^P) = \frac{E(Y^L)}{S_X(d)} = \frac{471.1954}{e^{-1/2}} = \boxed{776.8698}$$

ex3 math403 HW3 #3 Suppose the loss RV $X \sim U(0,100)$. An insurance policy for X has deductible $d = 20$ and a policy limit (max payment) of 60 per loss.

Find $E(Y^L)$.

Soln $Y^L = \begin{cases} 0 & X < 20 \\ X-20 & 20 < X < U = 80 = 60+20 \\ 60 & 80 < X < \underbrace{100}_{\text{support of } X = [0,100]} \end{cases}$

So $E(Y^L) = \int_{20}^{80} (x-20) f(x)\,dx + \int_{80}^{100} 60 f(x)\,dx$

$= \int_{20}^{80} (x-20) \frac{1}{100}\,dx + \int_{80}^{100} 60 \frac{1}{100}\,dx$

as in the HW problem.

4) LER and inflation $\qquad LER(d) = \dfrac{E(X_{new} \wedge d)}{E(X_{new})}$

$X_{new} = (1+r)X$

If $X \sim SF(\theta, \tau)$ then

$X_{new} \sim SF((1+r)\theta, \tau)$ so use formulas for

X_{new}. Also $(1+r)X \wedge d = (1+r)\,X \wedge \frac{(1+r)d}{1+r}$

$= (1+r)\left(X \wedge \frac{d}{1+r}\right) = (1+r)\min\left(X, \frac{d}{1+r}\right)$. So $LER_{new}(d)$

$= \frac{(1+r)E(X \wedge \frac{d}{1+r})}{(1+r)\,E(X)} = \frac{E(X \wedge \frac{d}{1+r})}{E(X)}$ M40 ~~88~~ 8.5

423 If $X \sim EXP(\theta)$,

$$LER(d) = 1 - e^{-d/\theta}.$$

If $X \sim$ (2 parameter) pareto $(\alpha > 1, \theta)$

$$LER(d) = 1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}.$$

If $X \sim$ single parameter $(\alpha > 1, \theta \le d)$,

$$LER(d) = 1 - \frac{(\theta/d)^{\alpha-1}}{\alpha}.$$

Ch9 Aggregate Loss Models

1] Once in a while Exam C has a question on a bonus or dividend. The loss ratio r or R is the proportion of aggregate losses to premiums. Express the bonus in terms of the earned premium, usually $\max(0, c(rP - X))$ where P is the earned premium, X the aggregate losses and $c > 0$ is

Fin rev 404 Final

For all know how to compute

$\bar{X}$ and $\hat{\sigma}^2_E$ for 1 sample.

Brand name
Discrete RVs: $X = N$ counts something

usually number of claims.

contin RV: X is usually loss amount

A) MLE I) Let $X_1, \ldots, X_n$ are iid Ber(θ)

$= \text{bin}(\theta, m=1)$ and $\sum_{i=1}^{n} X_i = \#$ 1s.

If $X_1 = x_1, \ldots, X_n = x_n$ $x_i \in \{0,1\}$ are

known, the likelihood is $L(\theta|\underline{x}) = \theta^{\sum x_i} (1-\theta)^{n - \sum x_i}$.

If only $\sum X_i = k$ is known, use the

bin ($\theta, m=n$) likelihood $L(\theta|k) = \binom{n}{k} \theta^k (1-\theta)^{n-k}$

II) (Positive) payments $= Y_i^P$

losses $= X_i$ usually left

truncated and often right censored.

deductible d

maximum payment $U-d$
maximum covered loss
$U = U-d+d$

47h) – 54) are important

$X = \frac{Y^{p}}{\alpha} + d$ where usually $\alpha = 1$

X is left truncated at d and right censored at U

If $n-k$ cases (losses) are uncensored and k are censored at U

$$L(\theta) = \frac{\left[\prod_{i=1}^{n-k} f(x_i)\right] \left[1 - F(U)\right]^{k}}{\left[1 - F(d)\right]^{n}}.$$

all n cases are truncated at d

MLE problem a) set-up likelihood
b) find MLE. b) often has easy formula in case a) was done wrong.

ex) C79 Losses come from a mixture of an EXP(100) dist with prob p and ano EXP (10000) dist with prob $1-p$. Losses of 100 and 2000 are observed. Find $L(p)$.

Soln) $f(x) = p\, f_1(x) + (1-p) f_2(x)$

$$= p \frac{1}{100} e^{-x/100} + (1-p) \frac{1}{10000} e^{-x/10000}$$

$f(x_1, x_2) = f(x_1) f(x_2) =$

$$\left(p \frac{1}{100} e^{-1} + (1-p) \frac{1}{10000} e^{-0.01}\right)\left(p \frac{1}{100} e^{-20} + (1-p) \frac{1}{10000} e^{-0.2}\right)$$

ex) C146) $X_1, \ldots, X_n$ are iid single parameter Pareto $(\alpha = \alpha_1, \theta = 1)$

ind < $Y_1, \ldots, Y_m$ are iid $(\alpha = \alpha_2, \theta = 1)$

$EY_2 = \frac{\alpha_2}{1-\alpha_2} = 1.5\, EY_1 = 1.5 \frac{\alpha_1}{1-\alpha_1}$

Want MLE for α_1 using both samples.
Find $\frac{d}{d\alpha_1} \log L(\alpha_1)$.

Soln $\alpha_2 = 1.5 \frac{\alpha_1}{1-\alpha_1} - \alpha_2\, 1.5 \frac{\alpha_1}{1-\alpha_1}$

$\alpha_2 \left(1 + 1.5 \frac{\alpha_1}{1-\alpha_1}\right) = \alpha_2 \frac{1+.5\alpha_1}{1-\alpha_1} = 1.5 \frac{\alpha_1}{1-\alpha_1}$ or $\alpha_2 = \frac{1.5\alpha_1}{1+.5\alpha_1} = \frac{3\alpha_1}{2+\alpha_1}$.

$$L = \prod_{i=1}^{n} \frac{\alpha_1}{x_i^{\alpha_1+1}} \prod_{i=1}^{m} \frac{\alpha_2}{y_i^{\alpha_2+1}} = \frac{\alpha_1^n}{\prod x_i^{\alpha_1+1}} \frac{\left(\frac{3\alpha_1}{2+\alpha_1}\right)^m}{\prod y_i^{\underbrace{\frac{3\alpha_1}{2+\alpha_1}+1}_{\frac{2+4\alpha_1}{2+\alpha_1}}}}$$

$$\ln L(\alpha_1) = n \ln \alpha_1 - (\alpha_1+1)\sum \ln x_i + m \ln\left(\frac{3\alpha_1}{2+\alpha_1}\right) - \left(\frac{2+4\alpha_1}{2+\alpha_1}\right)\sum_{i=1}^{m} \ln y_i$$

$$\frac{2+\alpha_1-\alpha_1}{\alpha_1(2+\alpha_1)}$$

$$\frac{\partial}{\partial \alpha_1} \ln L(\alpha_1) = \frac{n}{\alpha_1} - \sum_{i=1}^{n} \ln x_i + m \frac{2+\alpha_1}{3\alpha_1} \frac{(2+\alpha_1)3 - 3\alpha_1}{(2+\alpha_1)^2}$$

$$\frac{\partial n' - n\partial'}{\partial^2} \qquad - \frac{(2+\alpha_1)4 - (2+4\alpha_1)}{(2+\alpha_1)^2} \sum_{i=1}^{m} \ln y_i =$$

$$\boxed{\frac{n}{\alpha_1} - \sum_{i=1}^{n} \ln x_i + \frac{2m}{\alpha_1(2+\alpha_1)} - \frac{6}{(2+\alpha_1)^2} \sum_{i=1}^{m} \ln y_i \overset{\text{set}}{=} 0}$$

ex3C152 Observe losses 600, 700, 900.
No information is available about losses of 500 or less. Losses ~ EXP(θ). Find the MLE of θ.

Soln Losses are left truncated at $500 = d$.

$$\text{so } L(\theta) = \frac{\prod f(x_i)}{(1-F(500))^3} = \frac{\frac{1}{\theta}e^{-600/\theta}\,\frac{1}{\theta}e^{-700/\theta}\,\frac{1}{\theta}e^{-900/\theta}}{\left(e^{-500/\theta}\right)^3}$$

$$= \theta^{-3} e^{-(600+700+900-1500)/\theta} = \theta^{-3} e^{-700/\theta}$$

$$\ln L(\theta) = -3\ln\theta - \frac{700}{\theta}$$

$$\frac{d \ln L(\theta)}{d\theta} = \frac{-3}{\theta} + \frac{700}{\theta^2} \overset{\text{set}}{=} 0 \text{ or } 3\theta = 700 \text{ or } \hat{\theta} = \frac{700}{3}$$

Fred 3

(4) $f(x) = \dfrac{\alpha}{x^{\alpha+1}}$ $x > 1$

$\alpha > 0$.

observe 3 losses 3, 6, 14 and 2 losses above 25
$\underbrace{}_{\text{censored}}$

Find MLE of α.

Soln $L(\alpha) = \prod_{i=1}^{m} f(x_i) \; \big(\overbrace{1 - F(25)}^{\left(\frac{1}{25}\right)^{\alpha}}\big)^2$

$$= \frac{\alpha^3}{(3(6)(14))^{\alpha+1}} \frac{1}{(25)^{2\alpha}}$$

$\ln L(\alpha) = 3 \ln \alpha - (\alpha+1) \ln(3(6)(14)) - 2\alpha \ln 25$

$$\frac{d \ln L(\alpha)}{d\alpha} = \frac{3}{\alpha} - \ln[3(6)(14)] - 2\ln 25$$

$$= \frac{3}{\alpha} - 11.9672 \overset{\text{set}}{=} 0$$

$$\hat{\alpha} = \frac{3}{11.9672} = 0.2507$$

~~c(180) $n=2$ $\bar{X}=6$ x_1, x_2 iid $Exp(\theta)$~~

~~Find asymptotic variance of the MLE of $g(\theta) = 1 - e^{-10/\theta}(1+10\theta^{-1})$~~

~~Soln} $g'(\theta) = e^{-\frac{10}{\theta}} \frac{10}{\theta^2}(1+10\cdot\theta^{-1}) - e^{-\frac{10}{\theta}}(-10\theta^{-2})$~~

A good final would be all

E1: 2, E2: 1, 3, 4, 6 E3: 2, 4, 6

UIE2 1

ex} Simulate $\hat{\theta}$ 5 times: get

10, 28, 15 22, 18. If these values are equally likely and the true value of θ is 19, estimate bias.

Soln $\widehat{bias} = \frac{\sum_{i=1}^{B} \hat{\theta}_i}{B} - \theta = = \frac{93}{5} - 19 = -0.40$

$\widehat{MSE} = \frac{\sum (\hat{\theta}_i - \theta)^2}{B}$

no longer on final

Fred 4

ex) UIE2 14 There are 200 claims with aggregate loss 20000.

$X_i \sim Exp(\theta)$ $\quad$ $H_0\ \theta = 80$ ($Exp(80)$) $\quad$ $H_1\ \theta = 100$ ($Exp(\bar{x})$)

Find LRT test statistic T.

soln $n = 200$ $\quad \sum x_i = 20000 \quad \bar{X} = \hat{\theta}_{MLE} = 100$

$$L(\theta) = \prod_{i=1}^{200} \theta^{-1} e^{-x_i/\theta} = \theta^{-200} e^{-\sum x_i/\theta}$$

$$\ln L(\theta) = -200 \ln(\theta) - \frac{\sum x_i}{\theta}$$

$$\ln L_0 = -200 \ln(80) - \frac{20000}{80} = -1126.4053$$

$$\ln L_1 = -200 \ln(100) - \frac{20000}{100} = -1121.0340$$

$$T = 2(\ln L_1 - \ln L_0) = 10.7427$$

ex) UIE2 15

# claims	# obs's
0	6
1	0
2	4
3+	0

Above table is for a negative bin(r, β) data. Determine method of moments estimators of r and β then estimate the prob of one or more claims in a time period

$$\bar{X} = m = \frac{0(6)+2(4)}{10} = 0.8$$

$$\hat{\sigma}^2_E = t - m^2 = \frac{0^2(6)+2^2(4)}{10} - (.8)^2 = 0.96$$

$$\left.\begin{aligned} \hat{r}\hat{\beta} &\overset{set}{=} 0.8 \\ \hat{r}\hat{\beta}(1+\hat{\beta}) &\overset{set}{=} 0.96 \end{aligned}\right\} \quad \hat{r} = \frac{0.8}{\hat{\beta}}$$

$$1+\hat{\beta} = \frac{0.96}{0.8}$$

$$\hat{\beta} = 0.2$$

$$\hat{r} = \frac{.8}{.2} = 4$$

$$\widehat{P(N \geq 1)} = 1 - \hat{p}_0 = 1 - (1+\hat{\beta})^{-\hat{r}}$$

$$= 1 - (1.2)^{-4} = 0.518$$

$$\left(\hat{p}_0 = \binom{0+\hat{r}-1}{0}(1+\hat{\beta})^{-\hat{r}}\left(\frac{\hat{\beta}}{1+\hat{\beta}}\right)^0 = (1+\hat{\beta})^{-\hat{r}}\right)$$

ex C229 $\quad X_1, \dots, X_n$ iid with $f(x) = \dfrac{\theta}{(\theta+x)^2} \quad x>0 \quad \theta>0$ $\qquad$ Fred 5

Find the asymptotic variance of the MLE $\hat{\theta}$ of θ

Soln} $\quad \ln f(x) = \ln(\theta) - 2\ln(\theta+x)$

$$\frac{d \ln f(x)}{d\theta} = \frac{1}{\theta} - \frac{2}{\theta+x}$$

$$\frac{d^2 \ln f(x)}{d\theta^2} = \frac{-1}{\theta^2} + \frac{2}{(\theta+x)^2}$$

$$E\,\frac{d^2 \ln f(x)}{d\theta^2} = \frac{-1}{\theta^2} + \int_0^\infty \frac{2}{(\theta+x)^2}\,\frac{\theta}{(\theta+x)^2}\,dx$$

$$= \frac{-1}{\theta^2} + \int_0^\infty \frac{2\theta}{(\theta+x)^4}\,dx = \frac{-1}{\theta^2} + 2\theta \left.\frac{(\theta+x)^{-3}}{-3}\right|_0^\infty$$

$$= \frac{-1}{\theta^2} + \frac{2\theta}{3}\,\frac{1}{\theta^3} = \frac{-1}{\theta^2} + \frac{2}{3}\,\frac{1}{\theta^2} = \frac{-1}{3\theta^2}$$

$$I_1(\theta) = \frac{1}{3\theta^2}$$

$$V(\hat{\theta}) = \frac{1}{I_n(\theta)} = \frac{3\theta^2}{n}$$

C76 Bayesian posterior $x e^{-x}$

$N \sim Pois(\theta)$, θ has pdf ← dummy variable is θ

$\in G(\alpha'=2, \theta'=1)$

$f(\theta) = \theta e^{-\theta}$, $\theta > 0$ $\quad P_k = \frac{e^{-\theta}\theta^k}{k!}$

$$\int_0^\infty \theta e^{-n\theta} \, d\theta = \frac{1}{n^2}.$$

$$\pi(\theta | x = k) \propto \pi(\theta) f(k|\theta) \propto \theta e^{-\theta} e^{-\theta} \theta^k$$

$$= \theta^{\overbrace{k+1}^{\alpha-1=k+1}} e^{-2\theta} \quad \text{gamma}\left(\alpha^* = k+2, \theta^* = \tfrac{1}{2}\right)$$

$\alpha^* = \alpha' + k = k+2$ $\quad n=1$

$\theta^* = \frac{\theta'}{1+\theta'} = \frac{1}{1+1} = \frac{1}{2}$

1. A randomly selected policyholder had at least one claim last year.

want posterior prob this same policyholder will have at least one claim this year.

We will find the posterior pdf of

$$\pi(\theta | N > 0) \propto \pi(\theta) P(N > 0 | \theta)$$

$$\propto \theta e^{-\theta} \underbrace{(1 - e^{-\theta})}_{1-P_0}$$

$$\text{so } 1 = c \int_0^\infty \underset{\text{Gamma pdf}}{\theta e^{-\theta}} - \theta e^{-2\theta} \, d\theta = c - c\int_0^\infty \theta e^{-2\theta} d\theta$$

$$= c - c\frac{1}{4} = \frac{3c}{4} \quad \text{so } c = \frac{4}{3}$$

posterior pdf $= \frac{4}{3} \theta e^{-\theta}(1 - e^{-\theta})$

The final is a lot like the old final. 2 pages from Q11 4 pages from exams 1–3. 11 problems 12 parts 25 points each.

MLE problem: given $L(\theta)$. So easier than usual.

MME I like 23 i) iv) $G(\alpha, \theta)$ pareto(α, θ)

MME 45), 46