The method of moments estimator for $V(X)$ is the empirical variance 

$$\hat{\sigma}_E^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} X_i\right)^2$$

$$= \underline{t - m^2} \quad \text{(see 4)}.$$

3) The unbiased estimator of the variance is $\hat{\sigma}_U^2 = \frac{n}{n-1} \hat{\sigma}_E^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$,

= sample variance.

4) <u>Know</u> If there are 2 parameters $\gamma_1$ and $\gamma_2$, let $m = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$

t for two ⟷ m for mean

and $t = \frac{1}{n} \sum_{i=1}^{n} X_i^2$, the sample 2nd moment.

solving $E(X) \overset{set}{=} m$ and
$E(X^2) \overset{set}{=} t$ for $\gamma_1$ and $\gamma_2$

is equivalent to solving $E(X) \overset{set}{=} m$ and
$V(X) \overset{set}{=} \hat{\sigma}_E^2$ for $\gamma_1$ and $\gamma_2$;

both give the method of moments estimator. (for $\gamma = (\gamma_1, \gamma_2)$).

5] Unless told otherwise, use 1) or 4) to find the method of moments estimator. Occasionally, if $k=1$ ($\underset{\sim}{\gamma}$ is $k \times 1$) you are told to use $E(X^j) \overset{set}{=} \frac{1}{n}\sum x_i^j$ where $j$ is an integer, possibly $j=-1$ if $EX$ does not exist.
If $k=2$, sometimes you are told to use the $j$th moment and the $m$th moment where $\{j, m\} \neq \{1, 2\}$.
$\underbrace{\quad}_{d, k, h \text{ etc}}$

6] If $X \sim LN(\mu, \sigma)$, then $\log(X) = W \sim N(\mu, \sigma^2)$ and $X = e^W$.
$\mu = E(W) \neq E(X)$ and $\sigma^2 = V(W) \neq V(X)$.

ex] $X_1, \ldots, X_n \overset{iid}{\sim} EXP(\theta)$. Find the method of moments estimator of $\theta$.

Soln) $E(X) = \theta \overset{set}{=} \overline{X}$ so $\boxed{\hat{\theta} = \overline{X}}$.

7] * In general, ($K=1$) if $E(X) = g(\gamma)$

then $\hat{\gamma} = g^{-1}(\bar{x})$ if $g^{-1}$ exists

(eg $g\uparrow$ or $g\downarrow$).

proof: $E(X) = g(\gamma) \overset{set}{=} \bar{x}$

$$\hat{\gamma} = g^{-1}(\bar{x})$$

solve for $\gamma$ and put a hat on the solution.

ex] $X \sim G(\alpha, \theta)$. Find the method of moments estimator of $(\alpha, \theta)$.

Soln: $E(X) = \alpha\theta \overset{set}{=} m$ i)

$V(X) = \alpha\theta^2 \overset{set}{=} \hat{\sigma}_E^2$ ii)

$$\frac{V(X)}{E(X)} = \frac{ii)}{i)} = \boxed{\hat{\theta} = \frac{\hat{\sigma}_E^2}{m} = \frac{t - m^2}{m}}$$

$$\boxed{\hat{\alpha} = \frac{m}{\hat{\theta}} = \frac{m^2}{t - m^2}.}$$

Elrev see 23 i)

ex] $X \sim U(0, \theta)$. Find the method of moments estimator of $\theta$. (really $X_1, \ldots, X_n$ iid $U(0,\theta)$)

(7.5)

Soln: $E(X) = \frac{\theta}{2} \stackrel{set}{=} \overline{X}$

so $\boxed{\hat{\theta} = 2\overline{X}}$

really $X_1, \dots, X_n$ iid

ex] $X \sim$ Pareto $(\alpha, \theta)$. Find the MM estimators of $\alpha$ and $\theta$.

Soln: $E(X) = \frac{\theta}{\alpha - 1} \stackrel{set}{=} m$     i)

formula for $EX^k$

$$EX^2 = V(X) + (EX)^2 = \frac{\theta^2 \, \Gamma(\overset{k+1}{3}) \, \Gamma(\overset{\alpha-k}{\alpha - 2})}{\Gamma(\alpha)}$$

$$= \frac{\theta^2 \; 2 \; \Gamma(\alpha-2)}{(\alpha-1)(\alpha-2)\Gamma(\alpha-2)} = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} \stackrel{set}{=} t \quad \text{ii)}$$

$\Gamma(n) = (n-1)!$ so $\Gamma(3) = 2! = 2$

$\Gamma(x) = (x-1)\Gamma(x-1) = (x-2)(x-1)\Gamma(x-2)$

$$\frac{\text{ii)}}{[\text{i)}]^2} = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} \, \frac{(\alpha-1)^2}{\theta^2} = \frac{2(\alpha-1)}{\alpha-2} = \frac{t}{m^2}$$

$$2\alpha m^2 - 2m^2 = t\alpha - 2t$$

$$\alpha(2m^2 - t) = 2m^2 - 2t$$

M404 8

$$\hat{\alpha} = \frac{2m^2 - 2t}{2m^2 - t} = \frac{2(t - m^2)}{t - 2m^2} \quad \leftarrow \text{E1 rev 23 (U)}$$

Now calculate $\hat{\theta}$ by plugging $\hat{\alpha}$ into i):

$$\frac{\theta}{\alpha - 1} = m, \quad \hat{\theta} = m(\hat{\alpha} - 1)$$

but $$\hat{\alpha} - 1 = \frac{2t - 2m^2 - t + 2m^2}{t - 2m^2} = \frac{t}{t - 2m^2}.$$

so $$\hat{\theta} = \frac{mt}{t - 2m^2}. \quad \leftarrow \text{E1 rev 23 (U)}$$

ex] $X \sim LN(\mu, \sigma)$, $X_1, \ldots, X_n$ iid. Find the method of moments estimators of $\mu$ and $\sigma^2$.

Soln: $$E(X) = e^{\mu + 0.5\sigma^2} \overset{set}{=} m$$

$$E(X^k) = \exp\left(k\mu + \frac{1}{2}k^2\sigma^2\right)$$

APPA

(perhaps mgf of $\ln(x)$ evaluated at $t = 1$)

$$E(X^2) = \exp(2\mu + 2\sigma^2) \overset{set}{=} t$$

$$\mu + 0.5\sigma^2 = \ln(m) \qquad a)$$

$$2\mu + 2\sigma^2 = \ln(t) \qquad b)$$

6) - 2a) $= \hat{\sigma}^2 = \ln(t) - 2\ln(m)$ 8.5

$$\hat{\mu} = \ln(m) - 0.5\,\hat{\sigma}^2 = \ln(m) - \tfrac{1}{2}\ln(t) + \ln(m)$$
$$= 2\ln(m) - 0.5\ln(t)$$

several formulas E1 rev 23)

8} Given $\sum_{i=1}^{n} x_i$, $\sum_{i=1}^{n} x_i^2$ or $m = \bar{x}$ or $\hat{\sigma}_E^2$ or $t = \frac{1}{n}\sum x_i^2$ and $n$, find $\hat{\gamma}_1$ and $\hat{\gamma}_2$ and use these estimators to estimate some other quantity. See HW2, Q2.

like HW1 Q1

9} Let $\mu_j' = E(X^j) = \mu_j'(\gamma_1, \dots, \gamma_K)$.

Let $\hat{\mu}_j' = \frac{1}{n}\sum_{i=1}^{n} X_i^j$. Solve the system

$$\hat{\mu}_1' \stackrel{set}{=} \mu_1'(\gamma_1, \dots, \gamma_K)$$
$$\vdots$$
$$\hat{\mu}_K' \stackrel{set}{=} \mu_K'(\gamma_1, \dots, \gamma_K)$$

for the method of moments estimator

$\hat{\underline{\gamma}} = (\hat{\gamma}_1, \dots, \hat{\gamma}_K)$. 9} = 13 with different notation

10} If $g$ is a continuous function of the first $K$ moments and

$h(\underset{\sim}{\gamma}) = g(\mu_1'(\underset{\sim}{\gamma}), \ldots, \mu_k'(\underset{\sim}{\gamma}))$,

then the method of moments estimator

of $h(\underset{\sim}{\gamma})$ is $g(\hat{\mu}_1', \ldots, \hat{\mu}_k')$.

This property is similar to the invariance principle of the MLE.

ex] The method of moments estimator

of $V(X)$ is $\hat{\sigma}_E^2 = t - m^2 = \hat{\mu}_2' - (\hat{\mu}_1')^2$.

$V(X) = \mu_2'(\underset{\sim}{\gamma}) - [\mu_1'(\underset{\sim}{\gamma})]^2 = g(\mu_1', \mu_2')$.

So the method of moments estimator of $\quad g_2(\mu_1', \mu_2') = \sqrt{\mu_2' - (\mu_1')^2}$

$\sigma = SD(X)$ is $\sqrt{\hat{\sigma}_E^2}$.

$\leftarrow X_1 =$ point mass at 0 is common

11] Let $0 \le \varepsilon \le 1$, $F_{X_i}$, $f_{X_i}$, $p_{X_i}$ the cdf

pdf or pmf of $X_i$, $i = 1, 2$.

$X$ has a <u>mixture distribution</u> if

the cdf of $X$ is $F(x) = (1-\varepsilon) F_{X_1}(x) + \varepsilon F_{X_2}(x)$.

If $X_1$ and $X_2$ are continuous, the pdf of $X$

is $f(x) = (1-\varepsilon) f_{X_1}(x) + \varepsilon f_{X_2}(x)$, if both are

discrete, the pmf of $X$ is $p(x) = (1-\varepsilon) p_{X_1}(x) + \varepsilon p_{X_2}(x)$.

assume all expectations exist

$$E[g(X)] = (1-\varepsilon)\, E[g(X_1)] + \varepsilon\, E[g(X_2)]$$ (9.5)

$$E(X) = (1-\varepsilon)\, E(X_1) + \varepsilon\, E(X_2).$$

$$E(X^2) = (1-\varepsilon)\, E(X_1^2) + \varepsilon\, E(X_2^2) =$$

$$(1-\varepsilon)\left(V(X_1) + [E(X_1)]^2\right) + \varepsilon\left(V(X_2) + [E(X_2)]^2\right)$$

$$V(X) = E(X^2) - [E(X)]^2.$$

ex) $f(x) = \frac{w}{5} e^{-x/5} + \frac{1-w}{\theta} e^{-x/\theta}$

$X_1 \sim EXP(5)$, $X_2 \sim EXP(\theta)$, $w = \varepsilon$.

hard

$$E(X) = w5 + (1-w)\theta \overset{set}{=} m$$

$$E(X^2) = w(5^2+5^2) + (1-w)(\theta^2+\theta^2) \overset{set}{=} t$$

$$w(5-\theta) + \theta = m, \quad w = \frac{m-\theta}{5-\theta}$$

$$50\,\frac{m-\theta}{5-\theta} + \frac{5-\theta-m+\theta}{5-\theta}\, 2\theta^2 = t$$

$$50m - 50\theta + (5-m)\,2\theta^2 = 5t - \theta t$$

$$(5-m)\,2\theta^2 + (t-50)\theta + 50m - 5t = 0$$

$ax^2+bx+c$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\theta = \frac{50-t-\sqrt{(50-t)^2 - 4(5-m)2(50m-5t)}}{2(5-m)2}$$

$\theta$hat

If $m = 10$ and $t = 3000$, then (M404 10)

$$w = \frac{10-\theta}{5-\theta}, \quad -10\theta^2 + 2950\,\theta + \overbrace{500 - 15000}^{-14500} = 0$$

$$\theta^2 - 295\theta + 1450 = 0$$

$$\theta = \frac{295 \pm \sqrt{(295)^2 - 4(1450)}}{2} = \frac{295 \pm \sqrt{81225}}{2}$$

$$= \frac{295 \pm 285}{2} = 5 \text{ or } \boxed{290 = \hat{\theta}}$$

← can't get $w$ if $\theta = 5$

$$\hat{w} = \frac{10-290}{5-290} = \frac{-280}{-285} = \boxed{0.9825 = \hat{w}}$$

Note that if $\sum_{i=1}^{10} X_i = 100$ and $\sum_{i=1}^{10} X_i^2 = 30000$

then $m = \bar{X} = \frac{100}{10} = 10$ and $t = \frac{1}{10}\sum_{i=1}^{10} x_i^2 = \frac{30000}{10} = 3000.$

---

point mass at 0 or degenerate at 0

ex] Let $P(X_1 = 0) = 1$

| $x$ | $0$ |
|---|---|
| $P(X_1 = x)$ | $1$ |

used in zero inflated distributions

mine

So $E\,g(X_1) = \Sigma\, g(x)\,p(x) = g(0)(1) = g(0)$.

Let $F(x) = (1-\varepsilon)\, F_{X_1}(x) + \varepsilon\, F_{X_2}(x)$ where $X_2 \sim EXP(\theta)$.

$E(X) = (1-\varepsilon)\,0 + \varepsilon\,\theta \overset{set}{=} m$

$E(X^2) = (1-\varepsilon)\,0^2 + \varepsilon(\theta^2 + \theta^2) = 2\theta^2\varepsilon \overset{set}{=} t$

$\boxed{E X_2^2 = V(X_2) + [E X_2]^2}$

$$\frac{2\varepsilon\theta^2}{\varepsilon\theta} = 2\theta = \frac{t}{m}, \quad \boxed{\hat{\theta} = \frac{t}{2m}}$$



$$\varepsilon\hat{\theta} = m, \quad \hat{\varepsilon} = \frac{m}{\hat{\theta}} = \boxed{\frac{2m^2}{t}}.$$

( $\hat{\sigma}^2_E = t - m^2 \geq 0$ so $t \geq m^2 \geq 0$. Need $t \geq 2m^2$ for $\hat{\varepsilon} \in [0,1]$.)

If $\varepsilon$ was known, $\varepsilon\theta \overset{set}{=} m$ so $\hat{\theta} = \frac{m}{\varepsilon}$.

See HW 2 #1 for a similar problem.

12} Warning: for the bin$(q, m)$ distribution

m is the number of trials and

$$\hat{q} = \frac{\bar{X}}{m}$$

(using m as a parameter but also say $m = \bar{x}$)

$$E(X) = qm \overset{set}{=} \bar{x}$$

$$\hat{q} = \frac{\bar{x}}{m}$$

using bin$(q, k)$ with $\hat{q} = \frac{\bar{x}}{k} = \frac{m}{k}$

avoids confusion.

ex] $X \sim EXP(\theta)$, $\bar{x} = 22$ sample biased variance $= 361$. Fit the sample using the second moment and estimate the mean.

Soln} $\hat{\sigma}_E^2 = 361$, $\frac{1}{n}\sum x_i^2 = t = \hat{\sigma}_E^2 + m^2 = 361 + (22)^2 = 845 \overset{set}{=} 2\theta^2 = Ex^2$ so $\hat{\theta} = \sqrt{\frac{t}{2}} = \sqrt{\frac{845}{2}} = 20.5548 = \widehat{E(X)}$

ex} $X \sim$ inverse exponential $(\theta)$

data 25, 100, 200, 1000

Fit the data matching the $-1$ moment and estimate the mode $= \frac{\theta}{2}$.

Soln} $X \sim$ inverse weibull $(\theta, \tau = 1)$

$$E X^{-1} \underset{\substack{\uparrow \\ \text{exam1 rev}}}{=} \theta^{-1} \Gamma\left(1 - \frac{-1}{1}\right) = \theta^{-1} \underbrace{\Gamma(2)}_{1! = 1} = \theta^{-1}$$

$$\theta^{-1} = \frac{1}{\theta} \overset{set}{=} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i} = \frac{\frac{1}{25} + \frac{1}{100} + \frac{1}{200} + \frac{1}{1000}}{4}$$

$$\frac{1}{\theta} = 0.01400 \text{ or } \hat{\theta} = \frac{1}{0.014} = 71.4286$$

$$\text{so } \widehat{\text{mode}} = \frac{\hat{\theta}}{2} = \boxed{35.7143}$$

ex} Auto insurance has a deductible of 500. Claims for losses $< 500$ are not submitted. You observe a sample of 20 losses totalling 18000 (11.5)

$\underbrace{\text{including the deductible}}$

eg "loss 700" will have payment of $700-500=200$

The loss distribution is fitted to an exponential distribution using the method of moments. Estimate the average size of all losses including losses below the deductible.

Soln} This wording is confusing.

Want to estimate $E(Y^P) = e_X(d) =$

$\overset{=E(X)}{e(500) = \theta}$, the average payment (per positive payment).

So equate $\theta$ to the sample mean payment.

Since 18000 includes the deductible,

the sample mean payment is $\frac{18000}{20} - 500$

$= 400 \overset{set}{=} \theta$ So $\hat{\theta} = \widehat{E(Y^P)} = 400$.

(The payment is the loss for the insurance company, and does not include losses below the deductible, which are not submitted.)

cont skip

$X \sim NB(r, \beta)$ — method of moments 404

$n \, 11 \frac{3}{4}$

$E(X) = r\beta \overset{set}{=} m$

$V(X) = r\beta(1+\beta) \overset{set}{=} \hat{\sigma}_E^2 = t - m^2$

$$1+\beta = \frac{\hat{\sigma}_E^2}{m}$$

$$\hat{\beta} = \frac{\hat{\sigma}_E^2}{m} - 1 = \frac{\hat{\sigma}_E^2 - m}{m}$$

$$\hat{r} = \frac{m}{\hat{\beta}} = \frac{m^2}{\hat{\sigma}_E^2 - m}$$

$$\hat{\beta} = \frac{m}{\hat{r}}$$

---

skip R: $X \sim NB(n, p)$ has $p(x) = \frac{\Gamma(x+n)}{x!\,\Gamma(n)} p^n (1-p)^x, \; x = 0, 1, 2, \ldots$

APPA $p(x) = \frac{r(r+1)\cdots(r+x-1)}{x!} \frac{\beta^x}{(1+\beta)^{r+x}}$

so $r = n$, $p = \frac{1}{1+\beta}$ — size $= n = r$, prob $= p = \frac{1}{1+\beta}$

rnbinom(n=1000, size=10, prob=0.5) if $r = 10$, $\beta = 1$

ex) $\sum_{i=1}^{1000} X_i = 10097$ $\quad \sum X_i^2 = 123373$, $\quad n = 1000$

$m = 10.097$, $\quad t = \frac{1}{1000} \sum X_i^2 = 123.373$

end HW2 material

§ 15.1 13] Let $\underline{\gamma} = (\gamma_1, \dots, \gamma_K)$. <u>Percentile matching</u> matches K percentiles instead of K moments. Solve

$$\pi_{p_1}(\underline{\gamma}) \overset{set}{=} \hat{\pi}_{p_1}$$
$$\vdots$$
$$\pi_{p_K}(\underline{\gamma}) \overset{set}{=} \hat{\pi}_{p_K}$$

for $\hat{\gamma}_1, \dots, \hat{\gamma}_K$.

$F(\pi_{p_j}) = p_j$ and $F(\hat{\pi}_{p_j}) \approx p_j$.

$VaR_p(X) = \pi_p$ is given for some brand name distributions. Typically X is a continuous distribution.

14] Let $x_{(1)} \le x_{(2)} \le \dots \le x_{(n-1)} \le x_{(n)}$ be the order statistics of $x_1, \dots, x_n$

($x_{(1)}$ = min, $x_{(n)}$ = max)

ex] data

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1, & 7, & 3, & 8, & 6, & 4 \end{array}$$

$$\begin{array}{cccccc} 1, & 3, & 4, & 6, & 7, & 8 \\ x_{(1)} & x_{(2)} & x_{(3)} & x_{(4)} & x_{(5)} & x_{(6)} \end{array}$$

15] * The smoothed empirical estimator of a percentile $\pi_p$ is 12.5

$$\hat{\pi}_p = \begin{cases} (1-h)\, X_{(j)} + h\, X_{(j+1)}, & (n+1)p \text{ not an integer} \\ X_{(j)}, & j=(n+1)p \text{ an integer} \end{cases}$$

where $j = \lfloor (n+1)p \rfloor$ and $h = (n+1)p - j$.

$\lfloor 7.7 \rfloor = 7 = \lfloor 7 \rfloor$ $\qquad$ $\lfloor x \rfloor$ = greatest integer $\le x$

$\hat{\pi}_p$ is undefined if $j = 0$ or $n+1$. ($p < \frac{1}{n+1}$ or $p = 1$)

ex] ordered data 1, 2, 3, 4, 5, 6, 7, 8, 9

If $p = 0.9$, $(n+1)p = 10(0.9) = 9$

so $\hat{\pi}_{.9} = 9 = X_{(9)}$

If $p = .75$, $(n+1)p = \underbrace{7.5}_{\text{interpolate } X_{(7)} \text{ and } X_{(8)}}$, $j = 7$, $h = 0.5$ $= 7.5 - 7$

$\hat{\pi}_{.75} = 0.5\, X_{(7)} + 0.5\, X_{(8)} = \frac{7+8}{2} = 7.5.$

ex] $X \sim EXP(\theta)$. Find $\pi_p$ using $F(x)$. (often use VaRp(X))

Soln: $F(x) = 1 - e^{-x/\theta}$

$F(\pi_p) = 1 - e^{-\pi_p/\theta} = p$

or $1 - p = e^{-\pi_p/\theta}$

or $-\frac{\pi_p}{\theta} = \ln(1-p)$

or $\pi_p = \underbrace{-\theta \ln(1-p) = VaR_p(X)}_{\text{given on E1 rev}}$ *

16] one parameter: Given a table and a percentile, estimate $\theta$.

ex] C54 Losses follow an $EXP(\theta)$ dist. Find $\theta$ by matching the 80th percentile.

| interval | number (or proportion) | |
|---|---|---|
| [0, 100] | 32 | 80 |
| (100, 200] | 21 | |
| (200, 400] | 27 | |
| (400, 750] | 16 | 20 |
| (750, 1000] | 2 | |
| (1000, 1500) or ∞ | 2 | |
| | total = 100 | |

80 out of 100 $X_i \le 400$

So $F(400) \approx .8 = F(\pi_{.8})$

so $\hat{\pi}_{.8} = 400 \overset{set}{=} \pi_{.8} = VaR_{.8}(X)$

$$\hat{\pi}_{0.8} = 400 \overset{set}{=} \pi_{.8} = -\theta \ln(1-0.8) = -\theta \ln(0.2)$$

$$\hat{\theta} = \frac{-400}{\ln(0.2)} = \boxed{248.5340}$$

13.5

$\hat{\pi}_{.32} = 100$

$\hat{\pi}_{.53} = 200$

$\hat{\pi}_{.80} = 400$

$\hat{\pi}_{.96} = 750$

$\hat{\pi}_{.98} = 1000$

ex]

| interval | number |
|---|---|
| (0, 100] | 10 |
| (100, 200] | 10 |
| (200, 300] | 10 |
| (300, 400] | 10 |
| | 40 |

$\hat{\pi}_{.25 = \frac{10}{40}} = 100$

$\hat{\pi}_{.5 = \frac{20}{40}} = 200$

$\hat{\pi}_{.75} = 300$, $.75 = 30/40$

ex] $X \sim EXP(\theta)$

$$VaR_p(X) = \pi_p = -\theta \ln(1-p)$$

$$\hat{\theta} = \frac{\hat{\pi}_p}{-\ln(1-p)}$$