14] The mean and median of the posterior distribution are Bayesian analogs of point estimators.

15] It is possible to use an improper prior ($\pi(\underset{\sim}{\theta}) \geq 0$ is not a pdf or pmf because it does not integrate or sum to 1), but the posterior must be a pdf (or a pmf).

ex] $\pi(\theta) = \frac{1}{\theta}, \quad \theta > 0$ } improper pdfs
$\pi(\theta) = 1, \quad \theta \in \mathbb{R}$

16] The predictive density (pdf or pmf)

$$f(\underbrace{x_{n+1}}_{\text{often omit subscript}} \mid x_1, \ldots, x_n) = f(x_{n+1} \mid \underset{\sim}{x})$$

$$= \int f(x_{n+1} \mid \underset{\sim}{\theta})\, \pi(\underset{\sim}{\theta} \mid \underset{\sim}{x})\, d\underset{\sim}{\theta}$$

is the updated unconditional (marginal) pdf (or pmf) for X_{n+1} given the data $\underset{\sim}{x}$.

17] Although you can find $\underbrace{E(X_{n+1})}_{\text{bad notation}} =$ $E(X_{n+1}|\underline{x})$ using the predictive density $\left(E(X_{n+1}) = \int x f(x|\underline{x})\,dx\right)$, it is often easier to use $E(X_{n+1}) = E[X_{n+1}|\underline{x}]$ $= E\left[E(X_{n+1}|\Theta)|\underline{x}\right]$ = Bayesian Premium.

ex] Claim sizes follow a single parameter Pareto dist with $\alpha = 3$ and $\underset{RV}{\Theta}$ where $\Theta \sim U[1,4]$. An insured selected at random submits 4 claims of sizes 2, 3, 5, and 7.

Find i) the posterior mean of Θ

ii) the expected size of the next claim,

iii) the posterior prob that the next claim will be greater than 3,

iv) the posterior prob that the next claim will be less than 1.5.

soln]. $\pi(\theta) = \frac{1}{3}, \ 1 \le \theta \le 4$ $= \frac{1}{3} I(1 \le \theta \le 4)$

$$f(x|\theta) = \frac{3\theta^3}{x^4}, \ x \ge \theta \quad = \frac{3\theta^3}{x^4} I(x \ge \theta)$$

$$f(\underline{x}|\theta) = \prod_{i=1}^{4} f(x_i|\theta) \propto \theta^{12} I[x_{(1)} \geq \theta]$$
$$= \theta^{12}, \quad \theta \leq 2 \qquad = \theta^{12} I(\theta \leq 2)$$

$$\pi(\theta|\underline{x}) \propto \pi(\theta) f(\underline{x}|\theta) \propto \theta^{12} \; I(\theta \leq 2) \, I(1 \leq \theta \leq 4)$$
$$= \theta^{12} \; I(1 \leq \theta \leq 2)$$

$$\int_1^2 c\,\theta^{12} d\theta = c \frac{\theta^{13}}{13}\Big|_1^2 = c\left(\frac{2^{13}-1}{13}\right) \stackrel{set}{=} 1, \quad c = \frac{13}{2^{13}-1}.$$

$$\text{so } \pi(\theta|\underline{x}) = \frac{13}{2^{13}-1} \theta^{12}, \quad 1 \leq \theta \leq 2.$$

i) $$E(\Theta|\underline{x}) = \int \theta \, \pi(\theta|\underline{x}) \, d\theta = \int_1^2 \frac{13}{2^{13}-1} \theta^{13} d\theta$$

$$= \frac{13}{2^{13}-1} \frac{\theta^{14}}{14}\Big|_1^2 = \frac{13}{2^{13}-1} \frac{2^{14}-1}{14} = \boxed{1.8573}$$

ii) $$E[X_{n+1}|\underline{x}] = E\Big[\underbrace{E[X_{n+1}|\Theta]}_{\text{single parameter Pareto}(\alpha,\Theta)}\Big|\underline{x}\Big] = E\left[\frac{\alpha\Theta}{\alpha-1}\Big|\underline{x}\right] \underset{\alpha=3}{=} E[1.5\,\Theta|\underline{x}]$$

$$= 1.5\,E(\Theta|\underline{x}) \underset{\text{by i)}}{=} 1.5\,(1.8573) = \boxed{2.78595}$$

iii), iv) "next claim" means X_{n+1}

$$\text{Let } f(x_{n+1} \mid \underline{x}) = f(x \mid \underline{x}) = \int \underbrace{f(x \mid \theta)}_{\substack{x > \theta \\ \text{single parameter pareto}}} \; \underbrace{\pi(\theta \mid \underline{x})}_{\theta \in [1,2]} \, d\theta$$

$$= \int_1^{\min(2,x) = m} \frac{3\theta^3}{x^4} \; \frac{13}{2^{13}-1} \, \theta^{12} \, d\theta$$

{ bounds on integral $d\theta$: has $x > \theta$ and $\theta \in [1,2]$, so $1 \le \theta \le m$

$$= \frac{39}{2^{13}-1} \frac{1}{x^4} \int_1^m \theta^{15} d\theta = \frac{39}{2^{13}-1} \frac{1}{x^4} \frac{\theta^{16}}{16} \Big|_1^m$$

$$= \frac{39}{2^{13}-1} \frac{1}{x^4} \frac{m^{16}-1}{16}$$

$$= \begin{cases} \dfrac{39}{(2^{13}-1)16} \; \dfrac{x^{16}-1}{x^4}, & 1 \le x < 2 \\[2ex] \underbrace{\dfrac{39}{(2^{13}-1)16} \; \dfrac{2^{16}-1}{x^4}}, & x \ge 2 \end{cases}$$

iii) $$P(x_{n+1} > 3 \mid \underline{x}) = \frac{39(2^{16}-1)}{(2^{13}-1)16} \int_3^\infty x^{-4} \, dx$$

$$= \frac{39(2^{16}-1)}{(2^{13}-1)16} \frac{x^{-3}}{-3} \Big|_3^\infty = \frac{39(2^{16}-1)}{(2^{13}-1)16} \frac{1}{3(3^3)}$$

$$= \frac{19.5021}{3(3^3)} = \boxed{0.2408}$$

(v) $P(X_{n+1} \le 1.5 | \underline{x}) = \frac{39}{(2^{13}-1)16} \int_1^{1.5} x^{12} - x^{-4} \, dx$

do on HW 8

18] p407 The Bayesian estimator or Bayes estimator $\hat{\theta}$ minimizes the expected posterior loss function.

A) For the (mean) square error loss function, the Bayesian point estimator is the mean of the posterior distribution:

$\hat{\theta} = E(\Theta | \underline{x})$. $\quad \ell(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$.

A) is always used for credibility.

B) For the absolute value of the error loss function, the Bayesian point estimator is the median $\hat{\theta} = \pi_{0.5}$ of the posterior distribution. $\quad \ell(\hat{\theta}, \theta) = |\hat{\theta} - \theta|$.

C) For the zero one loss $\ell(\hat{\theta}, \theta) = \begin{cases} 0, & \theta = \hat{\theta} \\ 1, & \theta \neq \hat{\theta}, \end{cases}$ or $k \to$

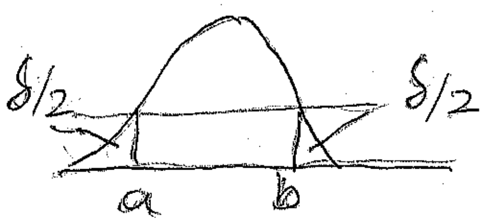
the Bayesian point estimator $\hat{\theta}$ = the mode of the posterior distribution.

19} PH10 Bayesian credibility intervals (a,b) 555

are the analogs of the CI where "credibility" has nothing to do with "actuarial credibility." want $\int_a^b \pi(\theta|\underline{X})\,d\theta \ge 1-\delta$.

The highest posterior density credibility set is the smallest set $(a_1,b_1) \cup \cdots \cup (a_k,b_k)$ where $\sum_{i=1}^{k} \int_{a_i}^{b_i} \pi(\theta|\underline{x})\,d\theta \ge 1-\delta$.

If the posterior is symmetric about the mode then $(a,b) = \left(\pi_{\frac{\delta}{2}}, \pi_{1-\frac{\delta}{2}}\right)$ = HPD $100(1-\delta)\%$ set = equal tail credibility interval.



Let $\hat{\theta}$ = posterior mean and $\hat{\sigma}$ = posterior SD = $\sqrt{\text{posterior variance}}$.

An approximate Bayesian $100(1-\delta)\%$ credibility region is $\hat{\theta} \pm \underbrace{z_{1-\frac{\delta}{2}}}_{1.645,\ 1.96,\ \text{or } 2.576} \hat{\sigma}$.

$\hat{\theta}$ and $\hat{\sigma}$ depend on n

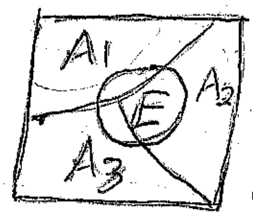
The Bayesian CLT says if $W_n = \theta | \underline{x}_n$, then $W_n \overset{approx}{\sim} N(\hat{\theta}, \hat{\sigma}^2)$ as $n \to \infty$, ie the

the posterior converges to a normal dist.

$W_n - X_n \xrightarrow{P} 0$
where $X_n \sim N(\theta, \frac{1}{n}\sigma^2)$

20] The sample space S is <u>partitioned</u> into n subsets $A_1, \dots, A_n$ if $P(A_i > 0)$, $A_i \cap A_j = \phi$ for $i \neq j$ and $S = A_1 \cup \dots \cup A_n$.

Then $E = \underbrace{(E \cap A_1) \cup \dots \cup (E \cap A_n)}_{\text{disjoint}}$



S

and $P(E) = \sum_{i=1}^{n} P(E \cap A_i) = \sum_{i=1}^{n} P(A_i) P(E | A_i)$.

21] Let $A_1, \dots, A_n$ partition S.

<u>Bayes rule</u>: $\underbrace{P(A_j | E)}_{\text{"posterior"}} = \dfrac{P(A_j \cap E)}{P(E)} = \dfrac{\overbrace{P(A_j)}^{\text{"prior"}} \overbrace{P(E|A_j)}^{\text{"likelihood"}}}{\underbrace{P(E)}_{\text{"marginal"}}}$

$$= \frac{P(A_j) P(E | A_j)}{P(A_1) P(E|A_1) + \dots + P(A_n) P(E | A_n)}.$$

If $n = 2$, $A_2 = \overline{A}$. so

$$P(A | E) = \frac{P(A) \; P(E|A)}{P(A) P(E|A) + P(\overline{A}) P(E | \overline{A})}.$$

ex) C136)

claim amount	Prob of claim class 1	amount class 2
250	0.5	0.7
2500	0.3	0.2
60000	0.2	0.1

$P(\text{class }1) = \frac{2}{3}$

$P(\text{class }2) = \frac{1}{3}$

Class 1 has twice as many claims as class 2. A claim of 250 is observed. Determine the Bayesian estimate of the expected value of the 2nd claim from the same policyholder.

soln: $P(\text{class }1 \mid \text{Claim} = 250) =$

$$\frac{P(\text{claim} = 250 \mid \text{Class }1)\, P(\text{class }1)}{P(\text{claim} = 250 \mid \text{class }1)\, P(\text{class }1) + P(\text{claim} = 250 \mid \text{class }2)\, P(\text{class }2)}$$

$$= \frac{0.5\left(\frac{2}{3}\right)}{0.5\left(\frac{2}{3}\right) + 0.7\left(\frac{1}{3}\right)} = \frac{10}{17}.$$ so $P(\text{class }2 \mid \text{claim} = 250) = \frac{7}{17}$.

$E[\text{claim} \mid \text{class }1] = .5(250) + .3(2500) + .2(60000) = 12875$

$E[\text{claim} \mid \text{class }2] = .7(250) + .2(2500) + .1(60000) = 6675$

"$E(\text{Claim} \mid 250)$" $= E(\text{claim} \mid \text{class }1)\, P(\text{class }1 \mid \text{claim} = 250)$

$+ E(\text{claim} \mid \text{class }2)\, P(\text{class }2 \mid \text{claim} = 250)$

$$= 12875\ \frac{10}{17} + 6675\ \frac{7}{17} = \boxed{10322}.$$

sum to 1