Math 480 HW9 2022. Due Wed. Nov. 9. Q9 on Wednesday Nov. 9 is on Poisson processes, nonhomogeneous Poisson processes, compound Poisson processes, iterated expectations and the conditional variance formula. See Exam 2 review 48), 56)-64) and HW8.

1) Let $T \sim$ gamma$(\alpha = 4, \lambda = 2)$.

a) We can interpret T as the waiting time for the _ _ _ _ _ _ _ th event (jump) of a Poisson process with rate _ _ _ _ _ _ _.

b) Find the mean and variance of T.

c) Let T_{100} be the sum of 100 iid random variables with the same distribution as T. Use the CLT (for the sum) to give a range where we expect T_{100} to lie with 95% probability.

Hint: Let $W = T_{100} = \sum_{i=1}^{100} T_i$ where $T_i \sim T$. Then $W \approx N(\sum_{i=1}^{100} E(T_i), \sum_{i=1}^{100} V(T_i))$. Do a backwards calculation where $P(-1.96 < Z < 1.96) = 0.95$ if $Z \sim N(0, 1)$.

d) Do you expect to see $T_{100} > 200$? Explain with the CLT. Hint: find $P(T_{100} > 200)$ using the CLT.

2) Recall that if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$,
then the ij entry of AB is found by matching the entries in the ith row of A with the jth column of B. The dimension of a matrix is $m \times n$ where m is the number of rows and n is the number of columns of the matrix. If the number of columns of A is equal to the number of rows of B then matrix multiplication is allowed (if the dimension of A is $m \times r$ and if the dimension of B is $r \times n$ then AB exists and is an $m \times r$ matrix. For the matrices A and B above, $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$.

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

If possible, find AB.

R Problems The R code for Math 480 homework is at (http://parker.ad.siu.edu/Olive/M480Rhw.txt).

3) Around 1995, there was a Markov chain program where a cartoon frog jumped from one of four lily pads to another. It took about 5 minutes for the frog to jump and we had to record about 20 jumps. Suppose the lily pads form a square with state 1 in the upper left corner, state 2 in the upper right corner, state 3 in the lower right corner and state 4 in the lower left corner. Assume the frog jumps at each time period, and jumps to the farthest corner (a diagonal jump) with probability 1/5 and to each of the other corners with probability 2/5. Then the transition matrix is

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0.4 & 0.2 & 0.4 \\ 0.4 & 0 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0 & 0.4 \\ 0.4 & 0.2 & 0.4 & 0 \end{bmatrix}.$$

a) Copy and paste the R commands for this part into R. This will give $\boldsymbol{C} = \boldsymbol{P}^2 = [P_{ij}^2]$. Write down $\boldsymbol{C}$.

b) $P^2_{11} = P(X_2 = 1|X_0 = 1) =$ P(frog is in state 1 at jump 2 | frog was in state 1 initially). What is P^2_{11}?

c) Copy and paste the R commands for this part into R. This will give $\boldsymbol{C} = \boldsymbol{P}^{40} = \boldsymbol{P}^{\infty} = [P^{\infty}_{ij}]$. Intuitively, by symmetry, the frog should be equally likely to be in each state after many jumps. Write down the ijth entry of $\boldsymbol{C} = \boldsymbol{P}^{40} = \boldsymbol{P}^{\infty}$.

4) This problem is like 3), but with

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0.4 & 0.2 & 0.4 \\ 0.4 & 0 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0 & 0.4 \\ 0.4 & 0.4 & 0.2 & 0 \end{bmatrix}.$$

Hence the frog jumps differently when in state 4.

a) Copy and paste the R commands for this part into R. This will give $\boldsymbol{C} = \boldsymbol{P}^2 = [P^2_{ij}]$. Write down $\boldsymbol{C}$.

b) $P^2_{11} = P(X_2 = 1|X_0 = 1) =$ P(frog is in state 1 at jump 2 | frog was in state 1 initially). What is P^2_{11}?

c) Copy and paste the R commands for this part into R. This will give $\boldsymbol{C} = \boldsymbol{P}^{1000} = \boldsymbol{P}^{\infty} = [P^{\infty}_{ij}]$. Write down the first row of $\boldsymbol{P}^{\infty}$.

Some good texts on linear algebra are various editions of

Anton, H., and Rorres, C. (1994), *Elementary Linear Algebra, Applications Version*, 7th ed., Wiley, New York, NY.

Leon, S.J. (1986), *Linear Algebra with Applications*, 2nd ed., Macmillan Publishing Company, New York, NY.

Carol Ash's website (https://faculty.math.illinois.edu/∼ash/) has good notes for advanced calculus, differential equations, discrete math, and linear algebra.