Math 480 Final, Fall 2018 Name____________________

1) Suppose that the joint pdf of the random variables Y_1 and Y_2 is given by

$$f(y_1, y_2) = \frac{2}{3}(2y_1 + y_2), \text{ if } 0 \le y_1 \le 1, 0 \le y_2 \le 1$$

and $f(y_1, y_2) = 0$, otherwise.

a) Find the marginal pdf of Y_1. Include the support.

$= \int f(y_1, y_2)\,dy_2 = \int_0^1 \frac{2}{3}(2y_1 + y_2)\,dy_2 = \frac{2}{3}\left(2y_1 y_2 + \frac{y_2^2}{2}\right)\Big|_0^1$

$$\boxed{= \frac{2}{3}\left(2y_1 + \frac{1}{2}\right) = \frac{1}{3}(1 + 4y_1) = \frac{4}{3}y_1 + \frac{1}{3},\ 0 \le y_1 \le 1}$$

b) Find $E(Y_1)$. $= \int_0^1 y_1 f_{Y_1}(y_1)\,dy_1 = \int_0^1 y_1 \left(\frac{4}{3}y_1 + \frac{1}{3}\right)dy_1$

$= \int_0^1 \frac{4}{3}y_1^2 + \frac{1}{3}y_1\,dy_1 = \left(\frac{4}{3}\frac{y_1^3}{3} + \frac{1}{3}\frac{y_1^2}{2}\right)\Big|_0^1 = \frac{4}{9} + \frac{1}{6} =$

$$\boxed{\frac{33}{54} = \frac{22}{36} = \frac{11}{18} = 0.6111}$$

c) Find $V(Y_1)$. $EY_1^2 = \int y_1^2 f_{Y_1}(y_1)\,dy_1 = \int_0^1 y_1^2\left(\frac{4}{3}y_1 + \frac{1}{3}\right)dy_1 = \int_0^1 \frac{4}{3}y_1^3 + \frac{1}{3}y_1^2\,dy_1$

$= \left(\frac{4}{3}\frac{y_1^4}{4} + \frac{1}{3}\frac{y_1^3}{3}\right)\Big|_0^1 = \frac{1}{3} + \frac{1}{9} = \frac{4}{9} = \underline{0.4444}$

$V(Y_1) = E(Y_1^2) - (EY_1)^2 = \frac{4}{9} - \left(\frac{11}{18}\right)^2 = \frac{23}{324} = \boxed{0.07099}$

d) Are Y_1 and Y_2 independent? Explain.

no $f(y_1, y_2) \ne g(y_1)\,h(y_2)$ on the support

60 $f_{Y_2}(y_2) = \frac{2}{3}(y_2 + 1)$ on $0 \le y_2 \le 1$

2) Let X be a random variable from a distribution with pdf

$$f(x) = \frac{\theta}{x^{1+\theta}}$$

where $x \geq 1$ and $\theta > 0$. Let $Y = X^2$ and find the pdf of Y using the method of transformations. Do not forget to include the support of Y. $x = (1, \infty)$

$y = t(x) = x^2$, $t(1) = 1$, $t(\infty) = \infty$ $y = (1, \infty)$

$x = \sqrt{y} = y^{\frac{1}{2}} = t^{-1}(y)$, $\left|\frac{d\, t^{-1}(y)}{dy}\right| = \left|\frac{1}{2} y^{-\frac{1}{2}}\right| = \frac{1}{2} y^{-\frac{1}{2}}$

$f_Y(y) = f_X(t^{-1}(y)) \left|\frac{d t^{-1}(y)}{dy}\right| = \frac{\theta}{(y^{\frac{1}{2}})^{1+\theta}} \frac{1}{2} y^{-\frac{1}{2}} = \frac{\theta}{2} \frac{1}{y^{\frac{1}{2}+\frac{\theta}{2}+\frac{1}{2}}}$

$$= \boxed{\frac{\theta}{2} \frac{1}{y^{1+\frac{\theta}{2}}} = \frac{\theta}{2} y^{-1-\frac{\theta}{2}}, \ y > 1}$$

$\frac{\theta}{2} \frac{1}{(\sqrt{y})^{2+\theta}} = \frac{\theta}{2} \frac{1}{(y^{\frac{1}{2}})^{2+\theta}} = \frac{\theta}{2} \frac{1}{y^{1+\frac{\theta}{2}}}$

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3) Suppose Y is the apple consumption of a randomly selected adult in 1987. Assume that the mean $\mu = 20.3$ and SD $\sigma = 5$ pounds per year. Assume that the sample mean $\overline{Y}$ is computed from a sample of size $n = 49$ and that the CLT holds. Find $P(\overline{Y} \geq 22)$.

$\mu_{\bar{Y}} = \mu = 20.3$ $\sigma_{\bar{Y}} = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{49}} = 0.7143$

20.3 22

$z = \frac{22 - 20.3}{0.7143} = 2.38$

	.08
2.3	.9913

$P(\bar{Y} \geq 22) = 1 - .9913 = \boxed{0.0087}$

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4) Suppose $X \sim$ Half Cauchy$(\mu = 0, \sigma = 1)$ with $F^{-1}(u) = \mu + \sigma \tan(\pi u/2)$. Simulate two values of x_i from this distribution if $u_1 = 0.95$ and $u_2 = 0.01$. (Hint: calculator should be in radians rad, not degrees deg, and $\tan(\pi/4) = 1$.)

$x_i = F^{-1}(u_i) = \tan(\pi u_i/2)$

$x_1 = \tan(\pi \cdot .95/2) = \boxed{12.7062}$

$x_2 = \tan(\pi \cdot .01/2) = \boxed{0.01571}$

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5) Suppose that the number of errors made by a Math 480 instructor for exam and quiz solutions is modelled by a Poisson process at a rate of 2 per hour. Find the probability of 7 errors if the instructor takes 3 hours to make the solutions in a semester.

$W \sim \text{poisson}(\lambda t = 2(3) = 6 = \mu)$

$$P(W=k) = \frac{e^{-\mu} \cdot \mu^k}{k!}$$

$$P(W=7) = \frac{e^{-6} 6^7}{7!} = \frac{693.8920}{5040} = \boxed{0.1377}$$

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$EY_i = \frac{\alpha}{\lambda} = \frac{8}{.02}$ $\quad VY_i = \frac{\alpha}{\lambda^2} = \frac{8}{(.02)^2} = 20000$

6) Suppose $X(t) = \sum_{i=1}^{N(t)} Y_i$ follows a compound Poisson process where the number of claims $N(t)$ occur at a rate of $\lambda = 5$ per day and the claim severity distribution Y_i is Gamma($\alpha = 8, \lambda = \beta = 0.02$) with $400 = E(Y_i)$.

a) Find $E[X(t)]$. $= \lambda t\, EY_1 = 5t\, 400 = \boxed{2000t}$

$V[X(t)] = \lambda t\,[V(Y_1) + (EY_1)^2]$

b) Find $SD[X(t)]$.

$= 5t\left[\frac{8}{(.02)^2} + (400)^2\right] = 5\,[20000 + (400)^2]\,t$

$SD(X(t)) = \sqrt{5[20000 + (400)^2]}\,\sqrt{t} = \boxed{948.6833\sqrt{t}}$

c) Find $E[N(14)]$, the expected number of claims in a 14 day period.

$N(14) \sim \text{Poisson}(\lambda t = 5(14) = 70)$

So $E[N(14)] = \boxed{70}$

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7) Suppose the number of claims per day $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process where $\lambda(t) = 5t$ and t is the time in days. Find the expected number of claims $E(N(7))$ in 7 days.

$m(7) = m(7) - m(0) = \int_0^7 \lambda(t)\,dt = \int_0^7 5t\,dt = 5\frac{t^2}{2}\Big|_0^7$

$= \frac{5}{2}\,49 = 122.5$

So $N(7) \sim \text{Poisson}(122.5)$

and $E[N(7)] = \boxed{122.5}$

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$$\boldsymbol{P} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \textit{not sent} & \textit{sent} & \textit{funded} \\ 0.6 & 0.25 & 0.15 \\ 0.2 & 0.6 & 0.2 \\ 0.05 & 0.65 & 0.3 \end{bmatrix} \end{array}.$$

3) DMS NSF grant proposals in Statistics are either 1) not sent, 2) sent for review but not funded, or 3) funded. Suppose a randomly selected researcher who submits such a grant proposal has the above transition matrix (where the time period is 1 year and $\boldsymbol{P}$ works for several years). Grant proposals are sent in October and early November. Consider researchers with proposals sent for review but not funded in 2018, find $\boldsymbol{\pi}_2$, the state vector for the researchers' proposals in 2020. M402 Q4d19

$\underline{\pi}_2 = \underline{\pi}_0 \mathbb{P}\mathbb{P} = [0\ 1\ 0]\mathbb{P}\mathbb{P} = [.2\quad .6\quad .2]\mathbb{P}$

$= .2(.6) + .6(.2) + .2(.05) = .25$

$.2(.25) + .6(.6) + .2(.65) = .54$

$.2(.15) + .6(.2) + .2(.3) = .21$

$= \boxed{[0.25 \quad 0.54 \quad 0.21]}$

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9) Suppose

$$\boldsymbol{P}^{(1)} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix} \end{array} \text{ and } \boldsymbol{P}^{(2)} = \begin{bmatrix} 0.3 & 0.7 \\ 0.9 & 0.1 \end{bmatrix}.$$

If the process begins in State 1, what is the probability that the process will be in State 2 after 2 steps? M402 Q4d19

$\underline{\pi}_2 = \underline{\pi}_0 \mathbb{P}^{(1)}\mathbb{P}^{(2)} = [1\ 0]\begin{bmatrix} .2 & .8 \\ .8 & .2 \end{bmatrix}\mathbb{P}^{(2)}$

$= [.2\quad .8]\begin{bmatrix} .3 & .7 \\ .9 & .1 \end{bmatrix} = [.2(.3) + .8(.9)\quad .2(.7) + .8(.1)]$

$= [0.78 \quad 0.22]$

so $\boxed{0.22}$

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10) Suppose that the joint pmf of Y_1 and Y_2 is $p(y_1, y_2)$ is tabled below.

$p(y_1, y_2)$		$y_2 = 0$	$y_2 = 1$	$y_2 = 2$	$P(Y_1 = y_1)$
y_1	0	1/9	2/9	1/9	4/9
	1	2/9	2/9	0/9	4/9
	2	1/9	0/9	0/9	1/9
$P(Y_2 = y_2)$		4/9	4/9	1/9	

$P(0,0) = \frac{1}{9} \neq P_{Y_1}(0) P_{Y_2}(0) = \frac{4}{9}\frac{4}{9} = \frac{16}{81}$

$P(2,2) = 0 \neq P_{Y_1}(2) P_{Y_2}(2) = \frac{1}{9}\frac{1}{9} = $

a) Are Y_1 and Y_2 independent? Explain.

no support is not a cross product

b) Find $E(Y_1)$. $= \sum y_1 P(y_1) = (0)\frac{4}{9} + (1)\frac{4}{9} + (2)\frac{1}{9} = \boxed{\frac{6}{9} = \frac{2}{3} = .6667}$

c) Find $E(Y_2)$. $= \sum y_2 P(y_2) = (0)\frac{4}{9} + (1)\frac{4}{9} + (2)\frac{1}{9} = \boxed{\frac{6}{9} = \frac{2}{3} = .6667}$

d) Find $\text{Cov}(Y_1, Y_2)$. $E Y_1 Y_2 = \sum \sum y_1 y_2 P(y_1, y_2)$

$= 0 + 0 + 0$
$+ 0 + (1)(1)\frac{2}{9} + 0$
$+ 0 + 0 + 0 = \frac{2}{9}$

$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$

$= \frac{2}{9} - \frac{2}{3}\frac{2}{3} = \frac{2-4}{9} = \boxed{\frac{-2}{9} = -0.2222}$

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$t=6 \quad \Delta = 1$

11) Suppose a stock price $A(t)$ follows an arithmetic Brownian motion with drift $\mu = 2$ and volatility $\sigma = 4$. If $A(6) = 12$, calculate the probability that $A(7) \leq 18$.

$w \sim A(t+\Delta)|A(t) \sim N(A(t) + \mu\Delta, \sigma^2\Delta) \sim N(12 + 2(1), 4^2(1))$

$w \sim N(14, 16), \quad P(w \leq 18)$

$z = \frac{14-18}{4} = 1.00$

$1.0 \,|\, .8413$

$P(w \leq 18) = \boxed{0.8413}$

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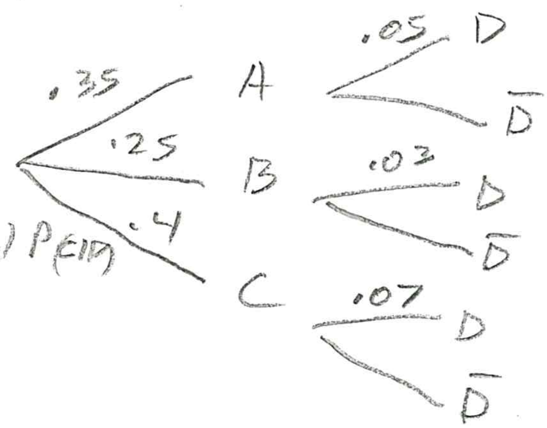
12) A company produces 1000 refrigerators a week at three plants. Plan A produces 350 refrigerators a week, plant B produces 250 refrigerators a week, and plant C produces 400 refrigerators a week. Production records indicate that 5% of the refrigerators produced at plant A will be defective, 3% of those produced at plant B will be defective, and 7% of those produced at plant C will be defective. All the refrigerators are shipped to a central warehouse. If a refrigerator at the warehouse is found to be defective, what is the probability that it was produced at plant A?

want $P(A|D)$



$P(D) = P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)$

$= P(A \cap D) + P(B \cap D) + P(C \cap D)$

$P(A \cap D) = P(A)P(D|A)$

So $P(A|D) = \frac{P(A \cap D)}{P(A \cap D) + P(B \cap D) + P(C \cap D)} = \frac{.35(.05)}{.35(.05) + .25(.03) + .4(.07)}$

$= \frac{.017500}{.05300} = \boxed{.3302}$

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