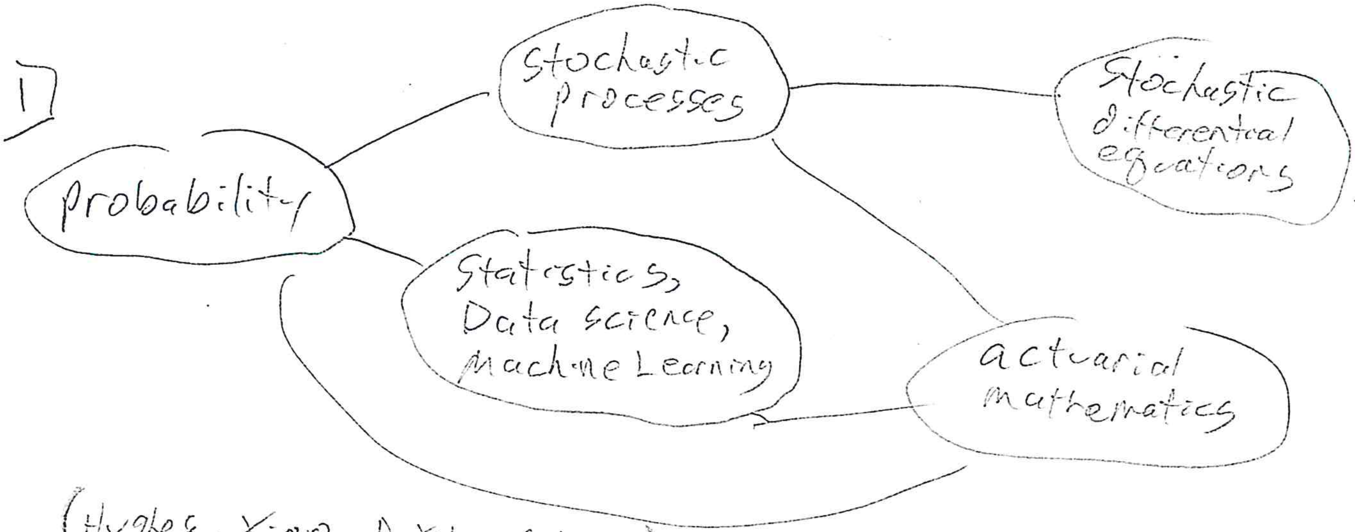
1]



(Hughes, Xiao, D. Xu, Schurz)

2] Probability and Stochastic processes are useful for many fields: financial mathematics (time series) [actuarial math], biology, ECE (signal and image processing), IE (reliability, queuing, geology (geostatistics, kriging), operations research, Statistics, etc.

3] stochastic process = random process = random field

4] A set is a collection of distinct elements.
Notation: use capital letters and braces
$A = \{1, 2, 3\}$.

5] * P.1 The sample space S is the set of all possible outcomes (sample points).

ex] Toss coin once $S = \{H, T\}$.
twice $S = \{HH, HT, TH, TT\}$.

ex] Machine makes 1000 items per day.
Draw every 100th item from production.

expt 1} check whether specs are met or not
$S_1 = \{$ 10 tuples each entry Y or N of the form
YYYNNNYNNY $\}$

(1.5)

expt 2} count how many of the 10 items selected met specs $S_2 = \{0, 1, \ldots, 10\}$.

6} * p. 2-3 An event is a subset of S. Event A occurs if an outcome of the experiment is in A.

7} * If every element in A is also in B, then A is a subset of B: $A \subset B$.

The empty set ϕ is the set with no elements.

8} * p3 The union of A and B is the set of all points in A or B or both!
$A \cup B = \{x \mid x$ is in A or B$\}$. $A \cup B$ occurs if either A or B occurs.
(or = inclusive or)

9} * p3 The intersection of A and B is the set of points in both A and B: $AB = A \cap B = \{x \mid x$ is in A and x is in B$\}$. $A \cap B$ occurs if both A and B occur.

10} * p. 3 If $A \subset S$ then the complement of $A = A^c = \bar{A}$, is the set of points in S but not in A:
$\bar{A} = A^c = \{x \in S \mid x \notin A\}$. $\bar{A}$ occurs if A does not occur

11} * p. 3 A and B are disjoint or mutually exclusive

12} Rules: List every element once, "order does not matter." i) $A \cup \bar{A} = S$

distributive laws ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3} De Morgan's Laws $\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

RHS: both $\bar{A}$ and $\bar{B}$ are used

If LHS has $\cap$, RHS has $\cup$.
$\cup$ $\cap$.

ex} $A = \{1,2\}$, $B = \{2,3\}$, $S = \{1,2,3,4,5\}$.
Then $A \subset S$, $\bar{A} = \{3,4,5\}$, $A \cap B = \{2\}$,
$A \cup B = \{1,2,3\}$.

14] An experiment is the process by which an observation is made. A sample point is a possible outcome from an experiment.

5} p.3 $\bigcup_{i=1}^{N} E_i$, $\bigcap_{i=1}^{N} E_i$, $\bigcap_{i=1}^{\infty} E_i$, $\bigcup_{i=1}^{\infty} E_i$ are used if there are N events or events $E_1, E_2, \ldots, E_N, \ldots$

16] p4 Probability is synonymous with chance, odds, and likelihood. The relative frequency interpretation of

Find the proportion of times each outcome occurs. Then the probability of outcome $E_1 = P(E_1)$ tends to the proportion of times E_1 would occur if the experiment was done infinitely often.

ex] toss (fair) coin, $S = \{H, T\}$, $P(H) = \frac{1}{2}$.

roll (fair) die, $P(5) = \frac{1}{6}$

17] KNOW For any event A, $0 \le P(A) \le 1$.

18] P4 The set valued function $P(\cdot)$ satisfies

axioms 1) $P(A) \ge 0$

2) $P(S) = 1$

3) If $A_1, A_2, \ldots$ are pairwise mutually exclusive events ($A_i \cap A_j = \emptyset$, $i \ne j$) then

$$P(A_1 \cup A_2 \cup \cdots) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Note] If $P(A) = 0$, A is impossible.

If $P(A) = 1$, A is certain to occur.

Hence $P(S) = 1$.

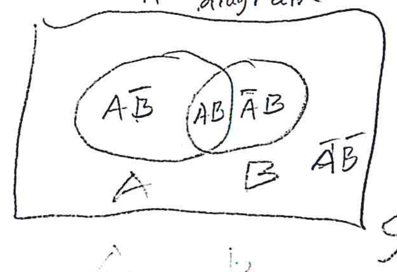
ex] roll die: $S = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{\text{die was } 7\}$. (not an event)

so $P(A) = 0$.

19] P4 * Complement rule: $P(A) = 1 - P(\bar{A})$.

General Addition rule

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$A\bar{B}$, AB and $\bar{A}B$ are pairwise mutually exclusive

$P(A) + P(B) = P(A\bar{B}) + P(AB) + P(AB) + P(\bar{A}B)$

21) * Addition rule for mutually exclusive events:

If $A_1, \ldots, A_n$ are pairwise mutually exclusive,

then $P(A_1 \cup A_2 \cup \cdots \cup A_n) = P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

$= P(A_1) + P(A_2) + \cdots + P(A_n)$.

22) A and $\bar{A}$ are mutually exclusive and $P(A \cup \bar{A}) = P(S) = 1$

so $P(A) = 1 - P(\bar{A})$ complement rule.

23) If $S = \{E_1, \ldots, E_k\}$ (elements), then $E_1, \ldots, E_k$ (sets)

are pairwise mutually exclusive.

24) * Addition rule for 2 mutually exclusive events:

If ($A \cap B = \emptyset$ so) A and B are mutually exclusive,

then $P(A \cup B) = P(A) + P(B)$

Use 21) with $A_1 = A$; $A_2 = B$, $n = 2$ and $P(\emptyset) = 0$,

or 20) with $P(A \cap B) = 0$.

25) know Common error! Student says $P(A \cup B) = P(A) + P(B)$

when $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Note) $S = S \cup \emptyset$, $S \cap \emptyset = \emptyset$ so $P(S) = P(S) + P(\emptyset)$ or $1 = 1 + 0$

so $P(\emptyset) = 0$.

26) Common problem: to list a sample space, use order (pg. 5)

ex) flip coin twice

TT
TH
HT
HH

3 times

T T T	H T T
T T H	H T H
T H T	H H T
T H H	H H H

outcomes equally likely for fair coin

ex) toss die twice

pr

1st \ 2nd	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2						
3						
4						
5						
6						

36 outcomes, equally likely for fair die

(5,4) — ↑ 1st, ↑ 2nd

27) *Finding the prob of an event via the sample point method! Suppose $S = \{E_1, \dots, E_k\}$.

Then $0 \le P(E_i) \le 1$ and $\sum_{i=1}^{k} P(E_i) = 1$.

↑ event $E_i = \{E_i\}$

An event A is a subset of S, so

$P(A) = \sum_{i: E_i \in A} P(E_i)$. That is, $P(A)$ is the sum of the probabilities of the sample points in A.

ex) Flip coin 3 times. Let A = 2 or more heads.

$P(A) = P(\{THH, HTH, HHT, HHH\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

for some outcomes blank,

ex) grade A B C D F (← mutually exclusive)
prob .2 .3 .2

Find prob of D or F $= P(D \cup F) =$

$1 - .2 - .3 - .2 = 0.3$

29] * multiplication principle: An experiment is performed in k parts $G_1, \dots, G_k$.
The number of possible outcomes for part $G_i = n_i$.
Then the total experiment consists of performing parts
G_1 then G_2 then ... then G_k and has

$\underbrace{n_1 \cdot n_2 \cdots n_k}_{\text{multiply}}$ possible outcomes.

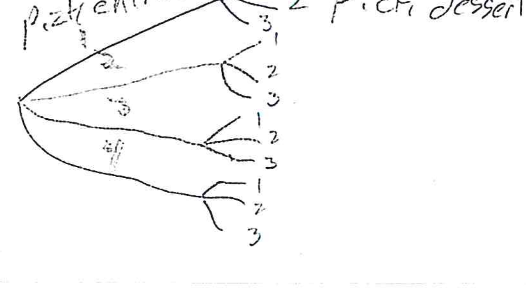
Technique 1: use slots to list the parts.

$\frac{n_1}{G_1} \quad \frac{n_2}{G_2} \quad \cdots \quad \frac{n_k}{G_k}$

Technique 2: For small experiments use tree diagrams.

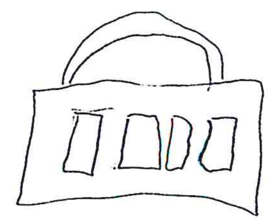
ex] A cafeteria offers 4 entrees and 3 desserts.
You may choose one entree and one dessert. How many
different meals can you choose?

Soln $\frac{4}{E} \cdot \frac{3}{D} = 12$ treediagram →



ex) A combination lock has 4 numbers, each with digits 0, ..., 9. How many combinations are possible? (4.5)

$$\frac{10}{1}\ \frac{10}{2}\ \frac{10}{3}\ \frac{10}{4} = \underbrace{10^4}_{\text{exponential notation}} = 10000$$



(artistically challenged)

30) An ordered arrangement of r distinct objects is a permutation. The number of ways of ordering n distinct objects taken r at a time is

$$P_r^n = \underset{1}{n}\ \underset{2}{(n-1)} \cdots \underset{r}{(n-r+1)} = \frac{n!}{(n-r)!},$$

$r-1$ objects have been used, $n-(r-1)$ remain

Here $n! = (n)(n-1)(n-2)\cdots(3)(2)(1)$

$$n! = (n)(n-1)! = (n)(n-1)(n-2)! = n(n-1)(n-2)(n-3)!, \text{ etc.}$$

$$3! = 3\cdot 2\cdot 1 = 6, \quad 1! = 1, \quad 0! = 1 \text{ by convention.}$$

ex) $P_n^n = n!$

ex) Consider an expt in which a person is selected from a pop of size n, a 2nd person is selected from the remaining $n-1$, and a 3rd person from the remaining $n-2$. This process is called sampling without replacement. Each sample of size r is a permutation.

ex) Select 3 students from a class of size 30 without replacement. How many samples are there?

Soln) $P_3^{30} = \underline{30}\cdot\underline{29}\cdot\underline{28} = 24360.$

Now suppose that an object is selected from a pop. of size n, then this object is placed back into the pop and another object selected (it is possible that the same object is selected again). This process is called <u>sampling with replacement</u>.

The number of possible samples of size r =

$$\underset{1}{n}\ \underset{2}{n}\cdots\underset{r}{n} = n^r$$ (not a permutation).

ex] Select 3 students from a class of size 30 with replacement. There are $(30)^3 = 27000$ possible samples.

ex] choosing officers: club has 25 members. want a president and a secretary. How many ways can these positions be filled?

Soln) $P_2^{25} = \underset{P}{25}\ \underset{S}{24} = 600$

(see next page)

31] * The number of unordered subsets of size r (chosen without replacement) from n objects is the binomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = C_r^n = \frac{P_r^n}{r!}$$ (n choose r)

when order is not important, the r selected (distinct) objects are called a <u>combination</u>.

Note:] When order is important, the r selected objects are called a permutation

32) ★ The binomial theorem says $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$

ex) $\binom{n}{0} = \frac{n!}{0!\,n!} = 1$ (number of ways to choose 0 people to come up and wave)

$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{1(n-1)!} = n$ (number of ways to choose 1 person to come up and wave)

$\binom{n}{r} = \binom{n}{n-r}$, $\binom{n}{n-1} = n$ (number of ways to have n-1 people come up and wave so choose 1 person not to come up)

$\binom{n}{n} = 1$ (# ways for all n to come up and wave)

ex) Select a committee of 5 members from 100 senators in $\binom{100}{5} = \frac{100!}{5!\,95!} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 (95!)}{5 \cdot 4 \cdot 3 \cdot 2 \ (95!)}$

$= 75\,287\,520$ ways

ex) A fair coin is tossed 10 times. What is the prob of obtaining 3 or fewer heads?

soln) Let $S = \{$sequences of 10 coin flips$\}$.

S contains $\underset{1st}{2} \cdots \underset{10th}{2}$ 2^{10} elements.

Let A be the event of 3 or fewer heads

and let $E_i = \{i$ heads in 10 tosses$\}$, $i = 0, 1, 2, \ldots, 10$.

Then $P(A) = P(E_0) + P(E_1) + P(E_2) + P(E_3) =$

$\frac{\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3}}{2^{10}} = \frac{1 + 10 + 45 + 120}{1024} = \frac{176}{1024} = 0.171$

52 cards in 4 suits: Hearts, Diamonds (red), clubs, Spades (black)
of 13 cards each 2, 3, ..., 10, Jack, Queen, King, Ace.
(Jack, Queen, King: face cards)

Suppose a person is dealt a 5 card hand.

What is the prob that all 5 cards are spades?

soln} $\frac{\binom{13}{5}}{\binom{52}{5}} = \frac{1287}{2598960} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} =$

$\frac{P_5^{13}}{P_5^{52}} = 0.0004952$

Note: If the number of elements in S is found without using order (denominator), then the number of elements in A (eg desired hands) should also be found without using order.

ex} Let A_2 be the set of 5 card hands that contain exactly 2 kings 2 queens and one jack

$P(A_2) = \frac{\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}}$ mult principle $\overline{2K}\ \overline{2Q}\ \overline{1J}$

ex) A lot consists of 100 fuses. 5 are chosen at random. If all 5 work, the lot is accepted. Assume 20 fuses will not work. What is the prob the soln) Let A be the event the lot is accepted.

$$P(A) = \frac{\binom{20}{0}\binom{80}{5}}{\binom{100}{5}} \approx 0.32$$

← # of samples of size 5 with 0 defectives!
mult princ.
0 def all work

← total # of samples of size 5

ex) A, B, C ← n=3
← r=2

	president p	secretary s	
1	A	B	AB
2	B	A	
3	A	C	AC
4	C	A	
5	B	C	BC
6	C	B	

→ combine each of the r! permutations of the same r members to get a combination of the r members in $C_r^n = \binom{n}{r} = \frac{P_r^n}{r!}$ ways

← permute each combination in r! ways to get $P_r^n = r!\,C_r^n = r!\binom{n}{r}$ permutations

$3 \cdot 2 = 6$ ways $= P_2^3 = \frac{3!}{(3-2)!}$

permutations: order matters

$\binom{3}{2} = \frac{3!}{2!1!} = 3$ ways

combinations: order does not matter

ex) Deal 2 cards. Find the # of ways that the 1st is a king and the 2nd is a king

3 kings left

$$\underset{1st}{\binom{4}{1}}\ \underset{2nd}{\binom{3}{1}} = P_2^4 = \underline{4} \cdot \underline{3}$$

33) Know p7 The conditional probability of event A given event B has occurred is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0.$$

34) Think of a conditional probability as an experiment with sample space B instead of S. (The exp't where only outcomes in B are of interest.)

ex) F2001 M150

Section	A	B	C	D	F	row total
M	11	8	3	1	2	25
all others	19	41	40	1	37	138
column total	30	49	43	2	39	163

(Columns A–F are grades. 163 ↑ grand total)

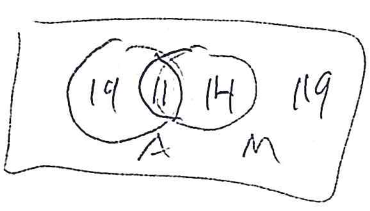
a) Find the prob that a randomly selected student got an A.

$$P(A) = \frac{30}{163} = 0.184$$

b) Find the prob that a randomly selected student got an A given the student was in section M

$$P(A|M) = \frac{P(A \cap M)}{P(M)} = \frac{\frac{11}{163}}{\frac{25}{163}} = \frac{11}{25} = \frac{11}{11+14} = 0.44$$

↑ row M acts as the new sample space



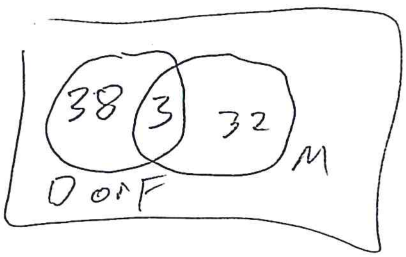
got a D or F.

75

$$P(D \cup F) = \frac{24+39}{163} = 0.251$$

d) Find the prob that a randomly selected student got a D or F given the student was in section M.

$$P(D \text{ or } F \mid M) = \frac{P(D \text{ or } F \cap M)}{P(M)} = \frac{\frac{3}{163}}{\frac{25}{163}} = \frac{3}{25} = 0.12$$



could have counts or proportions = probabilities

35} Common exam problem: You are given a table with i rows and j columns and asked to find conditional and unconditional probabilities.

Find the row, column, and grand totals and proceed as in the last ex.

Note: P(A) was unconditional while P(A|M) was conditional.

ex] $P(M \mid A) = \frac{11}{30} \approx .367$ ← col A acts as new sample space

$P(M) = \frac{25}{163} \approx .153$

P(A|M)	P(B|M)	P(C|M)	P(D|M)	P(F|M)	
$\frac{11}{25}$	$\frac{8}{25}$	$\frac{3}{25}$	$\frac{1}{25}$	$\frac{2}{25}$	} these