For a Markov Chain, $P(X_{n+1}=j \mid \text{past})$ depends only on the state the process was in at time $t=n$, not on $t<n$, so

Markov Property

$$P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, X_{n-2}=i_{n-2}, \ldots, X_1=i_1, X_0=i_0) \stackrel{\downarrow}{=}$$

$$P(X_{n+1}=j \mid X_n=i) = P_{ij}.$$

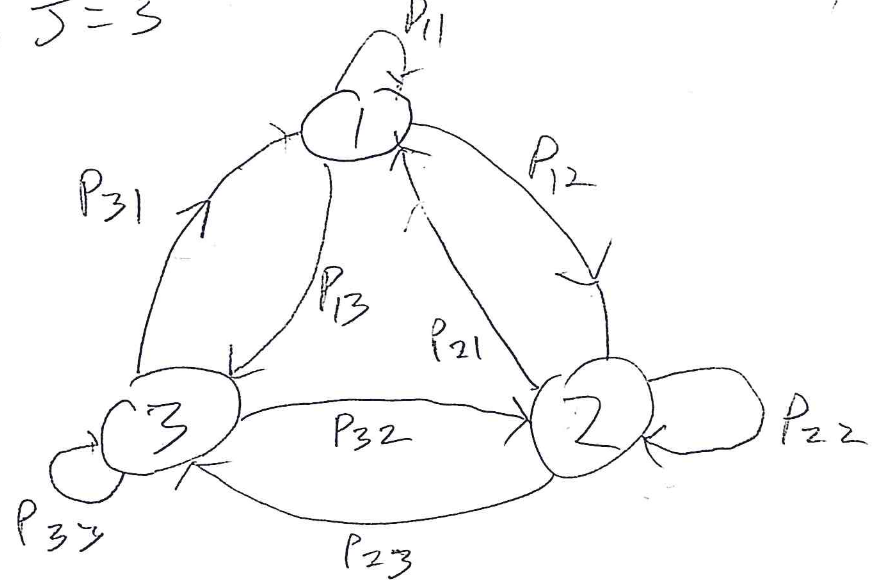
1} p183-184 The transition probability $P_{ij} = P(X_{n+1}=j \mid X_n=i)$

The transition probability matrix $\mathbb{P} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1J} \\ P_{21} & P_{22} & \cdots & P_{2J} \\ \vdots & & & \\ P_{J1} & P_{J2} & \cdots & P_{JJ} \end{bmatrix}.$

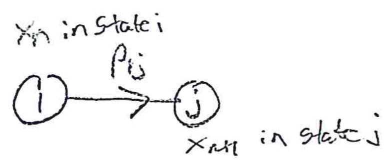
X_n in state i for ith row

X_{n+1} in state j for jth column

ex} $J=3$



transition diagram



2} p184 The sum of probs in any row of $\mathbb{P}$ is

$\sum_{j=1}^{J} P_{ij} = 1$ for row $i=1, \ldots, J$. Interpretation! Given $X_n = i$,

then X_{n+1} will be in state $j \in \{1, \ldots, J\}$ with prob 1.

P_{ij}^{n+m} ...

Let $\mathbb{P}^n = \underbrace{\mathbb{P} \cdots \mathbb{P}}_{\text{multiply } n \text{ times}}$. Then $\mathbb{P}^n = [P_{ij}^n]$ so P_{ij}^n is (56.5)

the ij th entry of $\mathbb{P}^n$. P_{ij}^n = prob that a process in state i will be in state j after n additional transitions. $\mathbb{P}^{n+m} = \mathbb{P}^n \mathbb{P}^m$ <u>Chapman-Kolmogorov equation</u>

§4.3 14} p194 <u>state j is accessible from state i</u>

if $P_{ij}^n > 0$ for some $n \geq 0$. State j is accessible from state i, iff, starting in state i, it is possible that the process will ever enter state j in a finite # of steps.

15} p195 Two states i and j that are accessible to each other <u>communicate</u>, written $i \leftrightarrow j$.

16} $P_{ii}^0 \overset{\text{def}}{=} P(X_0 = i \mid \underbrace{X_0 = i}_{\text{take } m=0}) = 1$

17} p195 i) State i communicates with state i, $i = 1, \ldots, J$

ii) If state i communicates with state j
then state j ... i.

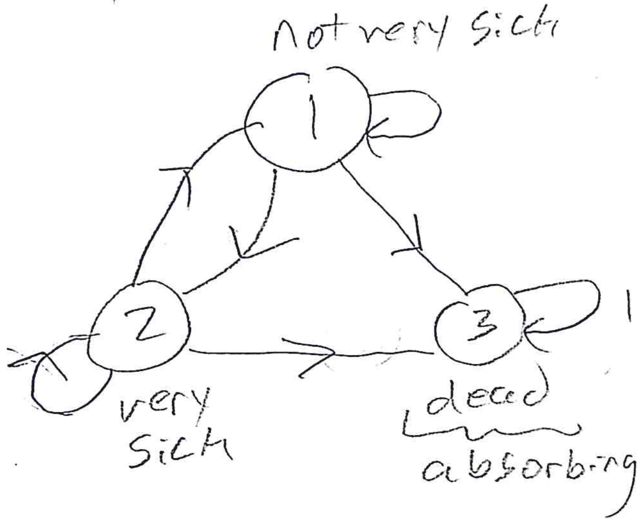
iii) If state i ... j
and state j ... k
then state i ... k.

... communicate with each ...
other form an equivalence class. A Markov Chain is irreducible if there is only one class, so all states communicate with each other.

19) p196 For state i, let r_i denote the prob, starting in state i, that the process will ever reenter state i. State i is recurrent if $r_i = 1$ and transient if $r_i < 1$. State i is absorbing if $P_{ii} = 1$. Once in an absorbing state, the Markov Chain stays in that state for all subsequent time periods.

An absorbing state is recurrent with $r_i = 1$, but not all recurrent states are absorbing.

ex)



20) In $\mathbb{P}$, an absorbing state has $P_{ii} = 1$, and all other entries $P_{ij} = 0$ in the ith row.

21) p196 A recurrent state will be revisited

infinitely often as $n \to \infty$. A transient state is not
certain to be revisited and will be visited only (575)
a finite # of times. Starting in a transient state
i, the # of time periods N the process will be in
state i, including the initial time, has a geometric
dist with finite mean $E(N) = \dfrac{1}{1-r_i}$.

2] For an irreducible Markov chain with J states,
all states are recurrent.

3] p.197 If state i is recurrent and $i \leftrightarrow j$, then state j is recurrent
transient transient,

4] state i is recurrent if $E(N) = \infty$
transient $E(N) < \infty$.

ex] $$\mathbb{P} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}$$

State 3 is absorbing, $1 \leftrightarrow 2$, $1 \not\leftrightarrow 3$, $2 \not\leftrightarrow 3$
$\{1, 2\}$ and $\{3\}$ are 2 classes, so $\mathbb{P}$ is not
irreducible.

ex) $$\mathbb{P} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.001 & 0.999 \end{bmatrix}$$ is irreducible

with class $\{1, 2, 3\}$.

one recurrent state in order for the process to continue for an indefinite length of time.

Hoel, Port, and Stone

26) * Let $\underset{\sim}{\pi}_n = (\pi_{1n}, \ldots, \pi_{Jn})$ represent the probabilities of being in states 1 to J at time n.

Let $\underset{\sim}{\pi}_0 = (\pi_{10}, \ldots, \pi_{J0})$ where $\pi_{i0} = P(X_0 = i) =$

P(process is in state i at the start). Note $\pi_{in} = P(X_n = i)$ for $i = 1, \ldots, J$.

ex) $\underset{\sim}{\pi}_0 = (0, \ldots, 0, \underset{\underset{j}{\uparrow}}{1}, 0, \ldots, 0)$ if process is certain to be in state j at time $n = 0$.

Then $\underset{\sim}{\pi}_n$ is the <u>state vector</u> at time n, and

$$\underset{\sim}{\pi}_n = \underset{\sim}{\pi}_0 \, \mathbb{P}^n = \underset{\sim}{\pi}_1 \, \mathbb{P}^{n-1} = \underset{\sim}{\pi}_2 \, \mathbb{P}^{n-2} = \underset{\sim}{\pi}_k \, \mathbb{P}^{n-k} = \underset{\sim}{\pi}_{n-1} \, \mathbb{P}.$$

$\underset{\sim}{\pi}_{n+1} = \underset{\sim}{\pi}_n \, \mathbb{P}$. Note that the $\underset{\sim}{\pi}_j$ are row vectors, $j \geq 0$.

$\underset{\sim}{\pi}_0$ is the <u>initial distribution</u> of the Markov chain.

27) * For a <u>nonhomogeneous</u> Markov chain, the matrix of transition probabilities $\mathbb{P}^{(k)} = \mathbb{P}_k$ depends on the kth step of the process. Then $\underset{\sim}{\pi}_n$ = state vector at time n satisfies

$\underline{\pi}_n = \underline{\pi}_0 \; P^{(1)} \; P^{(2)} \cdots P^{(n)}$

These need to be given.

58.5

Note that $P^n \neq P^{(n)}$.

28) * Typically the initial distr $\underline{\pi}_0 = (\pi_{10}, \ldots, \pi_{J0})$ is given, <u>or</u> you are told the chain is in state j so $\underline{\pi}_0 = (0, \ldots, 0, \underset{\substack{\uparrow \\ j}}{1}, 0, \ldots, 0)$.

ex) $P^{(1)} = \begin{bmatrix} .6 & .4 \\ .7 & .3 \end{bmatrix}$, $P^{(2)} = \begin{bmatrix} .5 & .5 \\ .8 & .2 \end{bmatrix}$.

If the process begins in state 2, what is the prob the process will be in state 1 after 2 steps.

<u>Soln</u> $\underline{\pi}_0 = (0, 1)$ and $\underline{\pi}_2 = (\pi_{12} \;\; \pi_{22})$, want π_{12}. Now $\underline{\pi}_2 = \underline{\pi}_0 \; P^{(1)} \; P^{(2)} =$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} .6 & .4 \\ .7 & .3 \end{bmatrix} \begin{bmatrix} .5 & .5 \\ .8 & .2 \end{bmatrix} = \begin{bmatrix} .7 & .3 \end{bmatrix} \begin{bmatrix} .5 & .5 \\ .8 & .2 \end{bmatrix} =$$

$$\begin{bmatrix} .35 + .24 & .35 + .06 \end{bmatrix} = \begin{bmatrix} .59 & .41 \end{bmatrix}. \text{ So } \pi_{12} = \boxed{0.59}.$$

(4) for a multistate model with ...

Healthy (0), Disabled (1), and Dead (2), for $k=0,1$,

transition prob matrix is

$p^{00}_{x+k} = 0.7$

$p^{01}_{x+k} = 0.2$

$p^{10}_{x+k} = 0.1$

$p^{12}_{x+k} = 0.25$

or

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.7 & 0.2 & 0.1 \\ 1 & 0.1 & 0.65 & 0.25 \\ 2 & 0 & 0 & 1 \end{array}$$

There are 100 lives at the start, all healthy. The future states are independent (for each life). Calculate the variance of the number of the original 100 lives who die within the first 2 years,

Soln} ways to go from (0) to (2) in 2 years

	prob
$0 \to 0 \to 2$	$.7(.1) = .07$
$0 \to 1 \to 2$	$.2(.25) = .05$
$0 \to 2$	$.1 = .1$
	0.22

↑ initial ↑ year 1 ↑ year 2

Let X = # who die in 2 years.

$X \sim \text{bin}(n=100, p=0.22)$,

$V(X) = np(1-p) = 100(0.22)0.78 = \boxed{17.16}$

You could also find $\overline{\pi_3} = \overline{\pi_0}\,\mathbb{P}\,\mathbb{P}$ " " → to get 0.22,

ex] A certain species of flower has 3 states: sustainable, endangered, and extinct. Transitions between states are modeled as a nonhomogeneous Markov chain with transition probability matrices

$$
\begin{array}{cccc}
 & \text{sustainable} & \text{endangered} & \text{extinct} \\
\mathcal{P}_0 = & & \begin{bmatrix} 0.85 & 0.15 & 0 \\ 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \end{bmatrix} &
\end{array}
$$

Start in endangered.

← For 1st transition, can only go to endangered or extinct

$$
\mathcal{P}_1 = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}
$$

For 2nd transition, if in endangered: can't get to extinct if state becomes sustainable, but can also go to extinct or endangered.

$$
\mathcal{P}_2 = \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}
$$

For 3rd transition if in endangered: can only get to extinct in 1 way.

$$
\mathcal{P}_n = \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad n = 3, 4, 5, \ldots
$$

← After 3rd transition, can't go extinct if sustainable or endangered.

Calculate the prob that a species endangered at time 0 will ever become extinct.

soln] The flower needs to go extinct in the 1st 3 transitions.

$$\underline{\pi}_3 = \underline{\pi}_0\, Q_0 Q_1 Q_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.85 & .15 & 0 \\ 0 & 0.7 & .3 \\ 0 & 0 & 1 \end{bmatrix} Q_1 Q_2$$

$$= \begin{bmatrix} 0 & 0.7 & 0.3 \end{bmatrix} \begin{bmatrix} x & x & x \\ 0.1 & 0.7 & 0.2 \\ 0 & 0 & 1 \end{bmatrix} Q_2 =$$

$$\begin{bmatrix} \underset{.7(.1)}{0.07} & \underset{.7(.7)}{0.49} & \underset{.7(.2)+.3}{0.44} \end{bmatrix} \begin{bmatrix} .95 & .05 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} .07(.95)+.49(.2) & .07(.05)+.49(.7) & 0+.49(.1)+.44 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1645 & 0.3465 & 0.4890 \end{bmatrix}$$

only need this

so $\boxed{0.4890}$

Or look at paths to extinct starting in Endangered at $n=0$

	prob
Ext	0.3
End → Ext	.7(.2) = 0.14
End → End → Ext	.7(.7)(.1) = 0.049
	0.489

$n=0$
transition 1 2 3

21) For a nonhomogeneous Markov chain, could have $\mathbb{P}^n \to \mathbb{P}^\infty$ or could have 60.5

$\mathbb{P}^n$ periodic so $\mathbb{P}^n$ takes on $K \geq 2$ matrices and does not converge. For an irreducible Markov chain, $\mathbb{P}^n \to \mathbb{P}^\infty$.

30) Let $C = \mathbb{P}^\infty$. Then $CC = C^2 = C$:

C is idempotent.

31) p220 If the homogeneous irreducible Markov chain is not periodic, then it is aperiodic and the limiting probability that the chain will be in state j is $\pi_{j,\infty}$ = long run proportion of time the chain is in state j.

$\underset{\sim}{\pi}_\infty = (\pi_{1\infty} \quad \pi_{2\infty} \quad \cdots \quad \pi_{J\infty})$. Also $\underset{\sim}{\pi}_\infty$ does not depend on the initial state X_0 of the chain.

$$\pi_{i\infty} = \lim_{n\to\infty} P(X_n = i), \; i = 1, \ldots, J, \text{ and } \underset{\sim}{\pi}_\infty = \lim_{n\to\infty} \underset{\sim}{\pi}_n.$$

Recall $\underset{\sim}{\pi}_n = (\pi_{1n}, \ldots, \pi_{Jn})$ where $\pi_{in} = P(X_n = i)$.