distribution of the irreducible aperiodic Markov Chain

Then $\lim_{n \to \infty} \mathbb{P}^n_{ij} = \pi_j \quad \forall i, j.$

So $\mathbb{P}^{\infty} = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_J \\ \pi_1 & \pi_2 & & \pi_J \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & & \pi_J \end{bmatrix} = \begin{bmatrix} \underline{\pi} \\ \underline{\pi} \\ \vdots \\ \underline{\pi} \end{bmatrix}$ each row $= \underline{\pi}$.

$\lim_{n \to \infty} \mathbb{P}^n = \mathbb{P}^{\infty}$

Note that $\underline{\pi}\, \mathbb{P}^{\infty} = [\pi_1 \sum_j \pi_j \quad \pi_2 \sum_j \pi_j \quad \cdots \quad \pi_J \sum_j \pi_j] = \underline{\pi}$.

It is also true that $\underline{\pi}\, \mathbb{P} = \underline{\pi}$, and $\pi_i > 0, i = 1, \ldots, J$

So $\pi_j = \pi_1 P_{1j} + \cdots + \pi_J P_{Jj} = \sum_{i=1}^{J} \pi_i P_{ij}$ for $j = 1, \ldots, J$.

See HW9 4C for such a $\mathbb{P}^{\infty}$.

$\mathbb{P}^T \underline{\pi}^T = 1\, \underline{\pi}^T$ or $\underline{\pi}^T$ is an eigenvector of $\mathbb{P}^T$ with eigenvalue 1.

ex) $\mathbb{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is irreducible but periodic.

$\mathbb{P}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So $\mathbb{P}^n = \begin{cases} \mathbb{P} & n \text{ odd} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & n \text{ even.} \end{cases}$

Skip §4.5–4.11. Ross also covers Markov chains that are not finite. Skip that material.

1) ~~you~~ Know The stochastic process $\{X(t), t \geq 0\}$ is a Brownian motion or Wiener process

if i) $X(0) = 0$

ii) the process has independent and stationary increments

iii) for every $t > 0$, $X(t) \sim N(0, \sigma^2 t)$,

Hence $E[X(t)] = 0$ and $V[X(t)] = \sigma^2 t \quad \forall t \geq 0$.

iv) $X(t)$ is continuous but nowhere differentiable in t.

(Norbert Wiener)

2) The sample paths of a Brownian motion are continuous but nowhere differentiable, hence the sample paths can't actually be drawn. Think of a falling leaf (hard to catch) or a tiny particle moving downwards slowly in a glass of water.

3) Know Let $0 \leq t_1 < t_2$. Then

$$X(t_2) - X(t_1) \sim N(0, \sigma^2 (t_2 - t_1)).$$

Let $t, s > 0$. Then $X(t+s) - X(s) \sim N(0, \sigma^2 t)$.

(Stationarity says the dist depends only on the length of the interval, so $X(t_2) - X(t_1) \sim X(t_2 - t_1)$ and
$X(t+s) - X(s) \sim X(t+s-s) = X(t)$.)

...) when ..., the process is a

standard Brownian motion. If $\{X(t), t \ge 0\}$ is a Brownian motion, then $\{Z(t), t \ge 0\}$ is a standard Brownian motion where $Z(t) = X(t)/\sigma$.

For $t, s \ge 0$, $Z(t) - Z(s) \sim N(0, |t-s|)$.

5) Let $W = Z(t+h) - Z(t)$, $h > 0$. Then $V\left(\frac{W}{h}\right) = V\left(\frac{Z(t+h)-Z(t)}{h}\right)$

$= \frac{1}{h^2} V(\underbrace{Z(t+h) - Z(t)}_{\text{length } h}) \underset{\sigma=1}{\overset{\uparrow}{=}} \frac{h}{h^2} = \frac{1}{h} \to \infty$ as $h \to 0$.

6) p 610 Let $0 < s < t$. Then $Z(s) \mid Z(t) = B \sim N\left[\frac{sB}{t}, \frac{s}{t}(t-s)\right]$

* Also $W = Z(t+s) \mid Z(t) \sim N(Z(t), s)$. where $E(W) = Z(t)$, $V(W) = s$.

$= [Z(t+s) - Z(t) + Z(t)] \mid Z(t)$

7) Let $0 = t_0 < t_1 < t_2 < \ldots < t_n$ where $t_i - t_{i-1} = h = \frac{100}{n}$ for $i = 1, \ldots, n$. Then $Z(t_n) = \sum_{i=1}^{n} [Z(t_i) - Z(t_{i-1})] = \sum_{i=1}^{n} W_i$

$= Z(t_1) - \underbrace{Z(t_0)}_{0} + Z(t_2) - Z(t_1) + \ldots + Z(t_n) - Z(t_{n-1})$

where $W_i = Z(t_i) - Z(t_{i-1})$ are iid $N(0, h)$. Let a vector

$B = \left(0, W_1, W_1 + W_2, \ldots, \sum_{i=1}^{j} W_i, \ldots, \sum_{i=1}^{n} W_i\right)$. A plot of $\left(0, \frac{100}{n}, \frac{200}{n}, \ldots, \frac{100 n}{n} = 100\right)$ vs B approximates a Brownian motion sample path on $[0, 100]$ as $n \to \infty$.

If $S_0 = 0$ and $S_j = \sum_{i=1}^{j} \underbrace{W_i}_{N(0,h)}$, then S_j is a random walk, see HW 10

... of a ...

The price of the stock at time 3 is 52. Determine the probability that the price of the stock is at least 55 at time 12. (62.5)

Soln} want $P(Z(12) > 55 \mid Z(3) = 52)$

$W \sim Z(12) \mid Z(3) = 52 \sim N(52, \sigma^2 = 12-3 = 9)$

So want $P(W > 55)$

$z = \frac{W\text{ value} - \mu_W}{\sigma_W} = \frac{55-52}{\sqrt{9}} = 1$

$P(W > 55) = P(Z > 1) = 1 - P(Z < 1.00)$

$= 1 - .8413 = \boxed{0.1587}$

$\begin{array}{c|c} & .00 \\ \hline 1.0 & .8413 \end{array}$

8) P612 A stochastic process $\{B(t), t \geq 0\}$ is a Brownian motion with drift coefficient μ and variance parameter σ^2 if

$B(t) = \mu t + \sigma Z(t)$ where $Z(t)$ is a standard Brownian motion. Then

i) $B(0) = 0$

ii) $B(t)$ has stationary and independent increments

iii) $B(t) \sim N(\mu t, \sigma^2 t)$.

10) Often we do not want $X(0) = 0$.

A stochastic process $\{A(t), t \geq 0\}$ is an <u>arithmetic Brownian motion</u> with drift coefficient μ and variance parameter σ^2 if $A(t) = A(0) + \mu t + \sigma Z(t)$ where $Z(t)$ is a standard Brownian motion.

$A(t) \sim N(A(0) + \mu t, \sigma^2 t)$. $A(t) - A(0) \sim N(\mu t, \sigma^2 t)$

$A(t+\Delta) - A(t) \sim N(\mu\Delta, \sigma^2\Delta)$

$A(t+\Delta) \mid A(t) \sim N(A(t) + \mu\Delta, \sigma^2\Delta)$, $t, \Delta > 0$.

11) Note that $B(t)$ is an arithmetic Brownian motion with $B(0) =$

Hence $B(t+\Delta) - B(t) \sim N(\mu\Delta, \sigma^2\Delta)$

$B(t+\Delta) \mid B(t) \sim N(B(t) + \mu\Delta, \sigma^2\Delta)$, $t, \Delta > 0$.

12) A stochastic process $\{G(t), t \geq 0\}$ is a geometric Brownian motion if

$\log(G(t)) = A(t)$.

$G(t) \sim \text{lognormal}(A(0) + \mu t, \sigma^2 t)$

$G(t)/G(0) \sim \text{lognormal}(\mu t, \sigma^2 t)$.

173 If $Y \sim N(\mu, \sigma^2)$ then

$X = e^Y \sim \text{lognormal}(\mu, \sigma^2)$.

$E(X^j) = E(e^{jY}) = \phi_Y(j)$ where $\phi_Y(t) = \exp(\mu t + \frac{\sigma^2}{2}t)$.

Hence $E(G(t)) = \phi_{A(t)}(1) \underset{\uparrow}{=} \exp(A(0) + \mu t, \frac{\sigma^2}{2}t)$

$A(t) = Y \sim \text{lognormal}(A(0) + \mu t, \sigma^2 t)$

$E\left(\frac{G(t)}{G(0)}\right) = \exp(\mu t + \frac{\sigma^2}{2}t)$.

Note that $\log\left(\frac{G(t)}{G(0)}\right) = \log(G(t)) - \underbrace{\log(G(0))}_{A(0)} \sim N(\mu t, \sigma^2 t)$.

Know for all

ex) Suppose the price of a stock B(t) follows an arithmetic Brownian motion with drift 1 and volatility 0.2. The current price of the stock is 40. Determine the prob that the price of the stock at time 4 is less than 43.

Let $A(t)$ = price of stock at time t. Let current time be t_0

Soln) $\mu = 1$, $\sigma = 0.2$.

$W = (A(t_0+4) \mid A(t_0) = 40) \sim N(40 + 1(4), (0.2)^2 4)$

$\Delta = 4$

$\sim N(44, (0.2)^2 4)$

	00
-2.5	.0062

want $P(W < 43)$.

$z^* = \frac{43-44}{0.2(2)} = \frac{-1}{0.4} = -2.5$

$P(W < 43) = P(Z < -2.50) = \boxed{0.0062}$

(63.5)

arithmetic Brownian motion with volatility 0.3. The current price of the stock is 52. The probability that the stock price is greater than 53 at time 0.5 is 0.2514. Determine the drift parameter.

Soln Take current time = 0 so $A(0) = 52$, where $A(t)$ = price of the stock at time t.

$$W = A(0.5) \mid A(0) = 52 \sim N\left(52 + \mu\, 0.5,\ (0.3)^2 0.5\right)$$

$$P(W > 53) = 0.2514 = P\left(Z > \frac{53 - 52 - \mu\, 0.5}{0.3\sqrt{0.5}}\right)$$

$= P(Z > z^*)$. So $P(Z \le z^*) = 1 - .2514 = .7486$

$$\begin{array}{c|c} & .07 \\ \hline 0.6 & .7486 \end{array}$$ so

$$\text{so } z^* = 0.67 = \frac{1 - 0.5\mu}{0.3\sqrt{0.5}} \quad \text{or } 1 - 0.5\mu = 0.67(0.3)\sqrt{0.5}$$

$$\text{or } \mu = 2\left(1 - .67(0.3)\sqrt{0.5}\right) = 2(.8579) = \boxed{1.7157}$$

Alternatively $A(t) = A(0) + \mu t + \sigma Z(t)$, so

$$W = A(0.5) = 52 + \mu\, 0.5 + 0.3\, \underline{Z(0.5)} \sim N\left(52 + \mu\, 0.5, (0.3)^2 0.5\right)$$

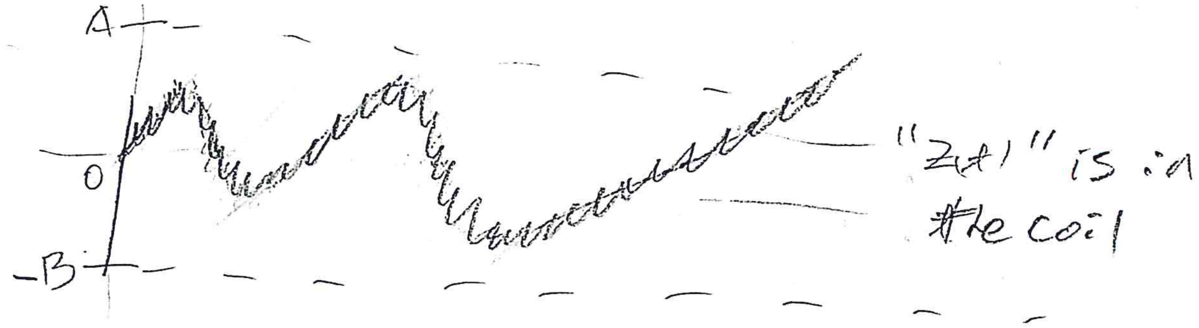
$$Z(t) \sim N(0, t) = N(0, 0.5)$$

and want $P(W > 53)$.

motion $Z(t)$. Then $P\left(Z(t) \geq A \text{ before } Z(t) \leq -B\right)$

$$= \frac{B}{A+B} \quad \text{where } A, B > 0.$$

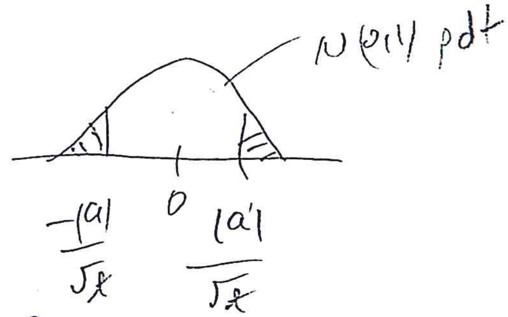
(64.5)



14) p 611–612 Let T_a denote the 1st time $Z(t)$ hits $a \in \mathbb{R}$. Then $P(T_a \leq t)$

$$= 2\,P\left(Z > \frac{|a|}{\sqrt{t}}\right) = 2\,P\left(Z \leq \frac{-|a|}{\sqrt{t}}\right).$$

where $Z \sim N(0,1)$.



Note $P(T_a \leq t) \to 1$ as $t \to \infty$. T_a is the hitting time!
the time $Z(t)$ first hits (equals) a.

For $a > 0$, $P\left[\max_{0 \leq s \leq t} Z(s) \geq a\right] = P(T_a \leq t)$

$$= 2\,P\left(Z \leq \frac{-a}{\sqrt{t}}\right).$$

Skip § 10.5, 10.6, 10.7

process $\{X(t), t \geq 0\}$ is a <u>stationary process</u> if for all n, s, the random vectors

$(X(t_1), \ldots, X(t_n))$ and $(X(t_1+s), \ldots, X(t_n+s))$

have the same distribution.

This condition is strong.

16] P.634 A stochastic process $\{X(t), t \geq 0\}$ is <u>second order stationary</u> or <u>weakly stationary</u> if $E[X(t)] = c$ and

$cov[X(t), X(t+s)] = R(s)$ do not depend on t.

Hence $cov(X(t), X(s))$ only depends on $|t-s|$.

ex] p630 Let $Z(t)$ be a standard Brownian motion and let $s \leq t$. Then $E[Z(t)] = 0 = c$

and $cov(Z(s), Z(t)) = cov[Z(s), Z(s) + \underbrace{Z(t) - Z(s)}_{\perp\!\!\!\perp\, Z(s)}]$

$= cov[Z(s), Z(s)] + \underbrace{cov(Z(s), Z(t) - Z(s))}_{0 \text{ by ind}}$

$= var(Z(s)) = s$. Similarly, $cov(Z(t), Z(t+s)) =$

$= \mathrm{cov}(Z(t), Z(t)) = t.$ So $Z(t)$ is not weakly stationary

In general, $\mathrm{cov}(Z(s), Z(t)) = \min(s, t)$ where $0 \le s, t$.

$\therefore Z(t)$ has stationary increments but $Z(t)$ is not stationary since $Z(t+s)$ has more variability than $Z(t)$.

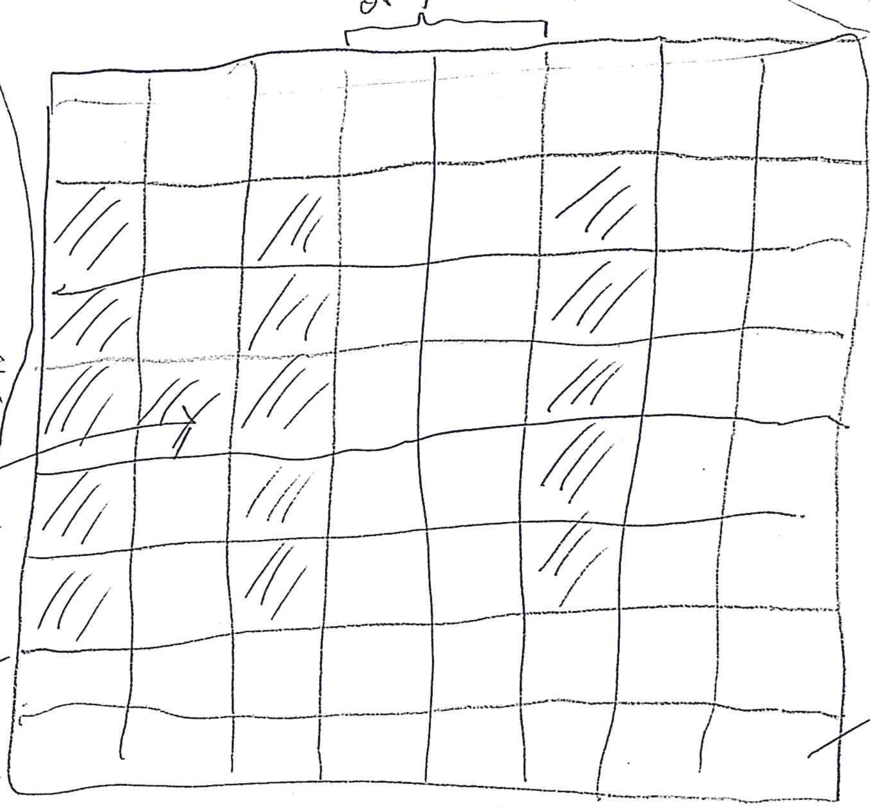
Skip §10.9 — go to notes 65.1

Engineering Applications of Probability

variants: more shades of white – black

ex) Image Processing

original always white



send 2 letters: could receive I I but a gap of 2 white would mean I I is an error

important pixel

always white

HI

image is often a picture of a famous scientist

pixel

64 pixels white or gray

1) Suppose $Y = \sum_{i=1}^{n} X_i$ where the X_i are iid and Y is discrete.

ex) $Y \sim bin(n, p)$, $Y = \sum_{i=1}^{n} W_i$ where $W_i \sim bin(1, p)$

ex) $Y \sim poisson(\lambda = n)$, $Y = \sum_{i=1}^{n} W_i$ where $W_i \sim poisson(1)$ $\frac{\lambda}{n}$

2) A normal approximation for Y uses
$X \sim N(\mu_Y, \sigma_Y^2)$ where $\mu_X = E X$ and $\sigma_X^2 = V X$,
$\mu_Y = E(Y) = n E(W_i)$ and $\sigma_Y^2 = V(Y) = n V(W_i)$.

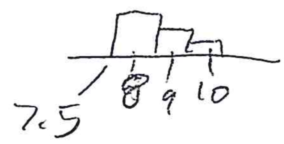
ex) $Y \sim bin(n, p)$ use $X \sim N(\mu = np, \sigma^2 = np(1-p))$.

ex) $Y \sim poisson(\lambda)$ use $X \sim N(\mu = \lambda, \sigma^2 = \lambda)$.

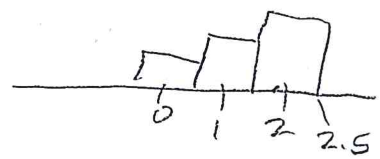
3) <u>Correction for continuity</u> When you use the normal approx for the binomial you are approximating the histogram corresponding to the pmf.
Let $Y \sim bin(n, p)$ and let y be an integer $\in \{0, n\}$.
Let $X \sim N(\mu = np, \sigma^2 = np(1-p))$ so $\sigma = \sqrt{np(1-p)}$.

$P(Y=y) \simeq P(y-\frac{1}{2} < y < y+\frac{1}{2})$ 9.5 10 10.5

ii) $P(Y \geq y) \simeq P(X \geq y - .5)$



iii) $P(Y \leq y) \simeq P(X \leq y + .5)$



know

4) Use the normal approx for the binomial

step *) check that $n > \frac{9p}{1-p}$ and $n > 9 \cdot \frac{1-p}{p}$

step 0) Find $\mu_x = np$ and $\sigma_x = \sqrt{np(1-p)}$

step i) Make the continuity correction and draw an X picture.

step ii) Z scores

iii) Z picture

(v) table

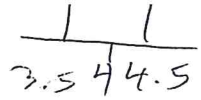
ex) $Y \sim bin(n=15, p=0.4)$.

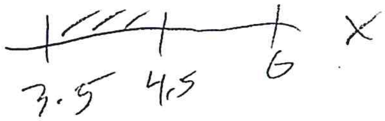
a) Find $P(Y=4)$ with the normal approx.

step *) $9\left(\frac{.4}{.6}\right) < 9\left(\frac{.6}{.4}\right) = 13.5 < n = 15$

step 0) $\mu_x = np = 15(.4) = 6$

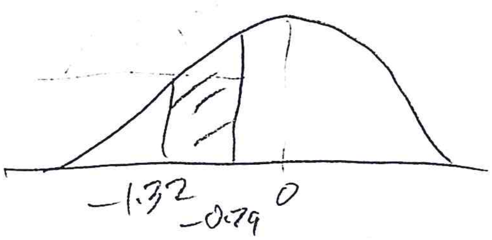
$\sigma_x = \sqrt{np(1-p)} = \sqrt{15(.4)(.6)} = \sqrt{3.6} = 1.8974$





ii) $\frac{3.5-6}{1.8974} = -1.32$, $\frac{4.5-6}{1.8974} = -0.79$

iii)



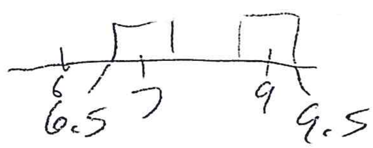
iv) $P(-1.32 < Z < -0.79) = P(Z < -0.79) - P(Z < -1.32)$

$= .2148 - .0934 = \boxed{.1214}$

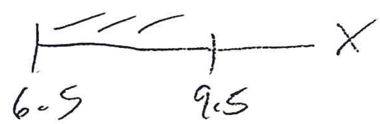
	02	09
-0.7	.2358	.2148
-1.3	.0934	

(exact value = .1268)

b) $P(7 \le Y \le 9)$

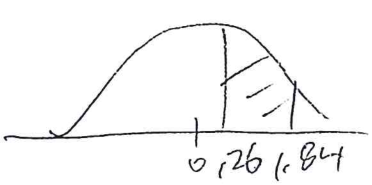


i)



ii) $\frac{6.5-6}{1.8974} = 0.26$ $\frac{9.5-6}{1.8974} = 1.84$

iii)



	04	06
0.2		.6026
1.8	.9671	

v) $P(Z < 1.84) - P(Z < 0.26) = .9671 - .6026 = \boxed{.3645}$

6.57

i) 

note: $P(7<Y) = P(8 \le Y) \approx P(7.5 < X)$.

ii) $z = \frac{6.5-6}{1.8974} = .26$

iii) 

iv) $P(z > .26) = 1 - P(z < .26) = 1 - .6026 = \underline{.3974}$

(This is easier than finding $\sum_{i=7}^{15} \underset{bin(15,.41)}{p(i)}$ or $1 - \sum_{i=0}^{6} \underset{bin(15,.41)}{p(i)}$.)

5) Warning: only apply the continuity correction when using a normal approx to a RV with a pmf usually binomial or poisson.

See HW 11 #3.