$x \leftarrow -\log(1-z)$

mean(x) = 1.0042

var(x) = 1.0269

§11.4

5) P.664-5 Inversion Method for a PMF with "support" $0, 1, \ldots, d, \ldots, J$ where $J = \infty$ is possible.

Let $U_{(n)}$ be the largest uniform pseudo RV given,

Suppose $F(d-1) < U_{(n)} \leq F(d)$,

k	$P_k = P(X=k)$	$F(k)$	range of U	resulting x_i
0	p_0	$p_0 = F(0)$	$0 \leq U < F(0)$	0
1	p_1	$F(0) + p_1 = F(1)$	$F(0) \leq U < F(1)$	1
$\vdots$		$\vdots$		
j	p_j	$F(j-1) + p_j = F(j)$	$F(j-1) \leq U < F(j)$	j
$\vdots$		$\vdots$		
d	p_d	$F(d-1) + p_d = F(d)$	$F(d-1) \leq U < F(d)$	d

ex simulate $X \sim$ poisson (2) if you have 4 $U(0,1)$ pseudo RVs

	0.65	0.12	0.48	0.95
x	2	0	2	3

$$P_k = \frac{e^{-\lambda} \lambda^k}{k!}$$

$P_1 = 2e^{-2} = .2707$ $F(1) = .4060 = F(0) + P_1$

$P_2 = \frac{2^2}{2} e^{-2} = .2707$ $F(2) = .6767 = F(1) + P_2$

$P_3 = \frac{2^3}{6} e^{-2} = .1804$ $F(3) = .8571 = F(2) + P_3$

$F(1) < .65 < F(2) \to x = 2$, $.12 < F(0) \to x = 0$

$F(1) < .48 < F(2) \to x = 2$, $F(2) < .85 < F(3) \to x = 3$

$e_1 \bmod e_2 = \underbrace{e_1 - \left\lfloor \frac{e_1}{e_2} \right\rfloor e_2}_{\text{remainder when dividing } e_1 \text{ by } e_2}$

eg $10 \bmod 7 = 3 = 10 - 1(7)$

p 646

16) U(0,1) random # generator $X_{n+1} = (aX_n + c)$ modulo m, X_0 = seed
$a = 69069$, $c = 1$, $m = 2^{32}$ is ok. then $U_i = \frac{X_i}{m}$, $i = 1$ by n.

Ch 7 Renewal Process

1) p409 Let $\{N(t), t \geq 0\}$ be a counting process, and let X_n be the interarrival (or waiting) time between the $(n-1)$th and nth events counted by the process, $n \geq 1$. If the nonnegative X_i are iid, then $\{N(t), t \geq 0\}$ is a renewal process.

ex) The renewal process is a Poisson process with rate λ if the X_i are iid EXP(λ).

2) p424, 409 $S_n = \sum_{i=1}^{n} X_i$ = time of occurence of nth event = waiting time until the nth event.

$S_n = S_{n-1} + X_n$ is a random walk with

$E(S_n) = n E(X_i)$ and $V(S_n) = n V(X_i)$.

$E(X_i) = \mu > 0$ since waiting times are nonnegative.

3) when an event occurs we say a renewal has taken place. Then S_n = time of nth renewal.

4) $N(t) < \infty$ $\forall t$, but $\lim_{t \to \infty} N(t) = \infty$.

§7.2

5) The number of renewals by time $t \geq n$ iff the nth renewal $\leq t$; $N(t) \geq n$ iff $S_n \leq t$.

$$P(N(t) = n) = P(N(t) \geq n) - P(N(t) \geq n+1)$$

$$= P(S_n \leq t) - P(S_{n+1} \leq t) = F_{S_n}(t) - F_{S_{n+1}}(t).$$

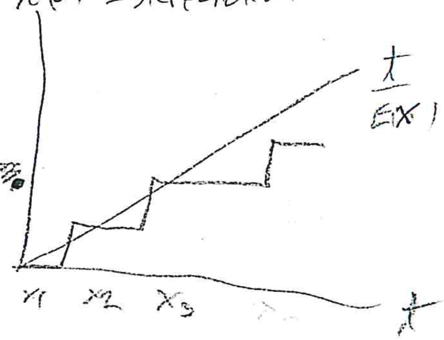
6) The dist of $S_n = \sum_{i=1}^{n} X_i$ for iid X_i is given in Exam 2 rev 38) for several brand name distr's.

ex) p 411 If the X_i are iid geometric(p)

then $\sum_{i=1}^{n} X_i \sim NB(n, p)$ negative binomial

7) If the $X_i \overset{iid}{\sim} EXP(\lambda)$, then $N(t) \sim Poisson(\lambda t)$.

§7.3 8) p415 $\frac{N(t)}{t} \to \frac{1}{E(X_i)} = \frac{1}{\mu}$ as $t \to \infty$. Hence about $\frac{t}{E(X_i)}$ ~~ab~~ of $N(t)$ call



70.5

8) p416 Let $\mu = E(X_i)$, Then $\frac{1}{\mu}$ = rate of the renewal process. Since the ave time between renewals is μ, the ave rate of renewal is 1 every μ time units.

9) p421 If the X_i are the interarrival times of a renewal process $N(t)$,

then $E\left[X_1 + \cdots + X_{N(t)+1}\right] = E(X)\, E[N(t)+1]$.

p419 Elementary Renewal Theorem:

10) Let $m(t) = E(N(t))$ = renewal function = meanvalue function,

Then $\frac{m(t)}{t} = \frac{E[N(t)]}{t} \to \frac{1}{\mu} = \frac{1}{E(X)}$ as $t \to \infty$.

11) Normal approximation (CLT for a renewal process)

Let $E(X) = \mu$ and $V(X) = \sigma^2$.

$$\frac{N(t) - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} \xrightarrow{D} N(0,1), \quad \text{as } t \to \infty.$$

$$\text{so } N(t) \approx N\left(\frac{t}{\mu}, \frac{t\sigma^2}{\mu^3}\right).$$

ex) $N(t)$ a poisson process so $X_i \sim Exp(\lambda)$, $\mu = \frac{1}{\lambda}$, $\sigma^2 = \frac{1}{\lambda^2}$

$N(t) \sim Poisson(\lambda t)$ so $m(t) = \lambda t$

and $\frac{\lambda t}{t} = \lambda \to \lambda = \frac{1}{\mu} = \frac{1}{\frac{1}{\lambda}}$.

$N(t) \approx N(\lambda t, \lambda t)$ (normal approx for Poisson(λt))

$\mu \quad \frac{1}{\lambda}$

since $\frac{t}{\mu} = \frac{t}{1/\lambda} = \lambda t$ $\qquad \frac{t}{\mu^3} = \frac{t/\lambda^2}{1/\lambda^3} = \lambda t,$

want $\lambda t \geq 9$ to use the normal approx.

See E3 rev 96}.

12} p 411–412 Let $W_1, W_2, \ldots$ be iid Bernoulli(p) ~ bin(1, p). $W_i = \begin{cases} 1 & \text{if } S \\ 0 & \text{if } F \end{cases}$

Let the <u>Bernoulli process</u> $N(t)$ count # S's = # 1's in $[0,t]$. The waiting times $X_i \overset{iid}{\sim}$ geom(p) = waiting time from (i−1)th S to ith S. $S_n = \sum_{i=1}^{n} X_i \sim NB(n, p) =$ waiting time until nth success with $S_0 = 0$.

The # 1's = sum of 1's = sum of W_i's in interval $[0,t]$.

So $N(t) = \sum_{i=1}^{\lfloor t \rfloor} W_i \sim \text{bin}(\lfloor t \rfloor, p)$ for $t \geq 1$.

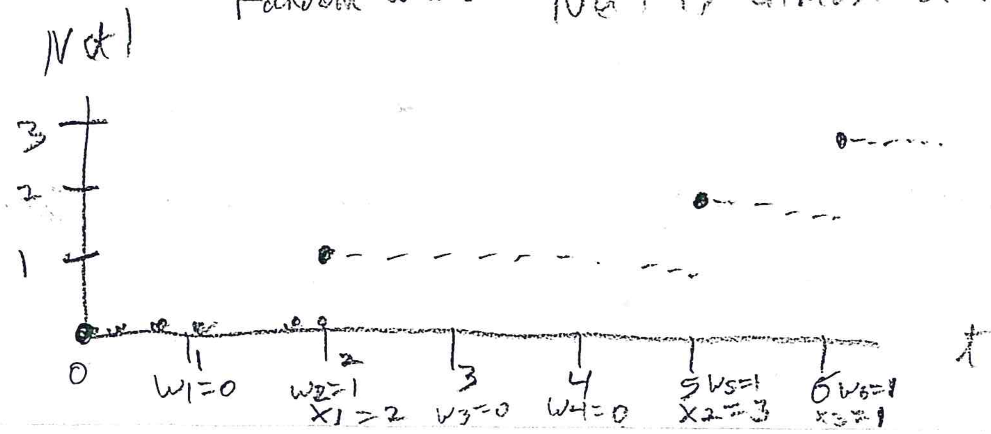
$N(t) = 0$ for $0 \leq t < 1$.

Think of 0, 1, 2, ... as units of time eg hours, minutes, seconds.

Let $N(j) = T_j = \underbrace{\sum_{i=1}^{j} W_i}_{\text{random walk}} \sim \text{bin}(j, p)$ with $T_0 = 0$ for $j \in \{1, 2, \ldots\}$.

$N(t)$ is almost a random walk T_j.

Sample path



S's can only occur at times $t = 1, 2, 3, \ldots$, More results on probability.

1) p51-52, 95 See E3 rev 109).

Suppose there is a finite population of size N. C objects are of the 1st type (that you want to count or "success") and N-C of the second type. Suppose n objects are selected at random without replacement from the N objects. Let Y count the number of n selected objects that were of the 1st type. Y has a hypergeometric dist, $Y \sim HG(C, N-C, n)$ with pmf

$$p(y) = \frac{\binom{C}{y}\binom{N-C}{n-y}}{\binom{N}{n}}$$ where the integer y

satisfies $\max(0, n-N+C) \le y \le \min(n, C)$.

Let $p = \frac{C}{N}$. Then $E(Y) = \frac{nC}{N} = np$ and

$$V(Y) = np(1-p)\frac{N-n}{N-1}.$$

2) If the sampling was with replacement $Y \sim bin(n, p)$.

ex) pop of N voters survey!. C will

4) a) If $n << c$ and $n << N-c$, then
$Y \approx \text{bin}(n, p)$.

b) If n is large and a) holds, then

$$Y \approx N(np,\ np(1-p)).$$

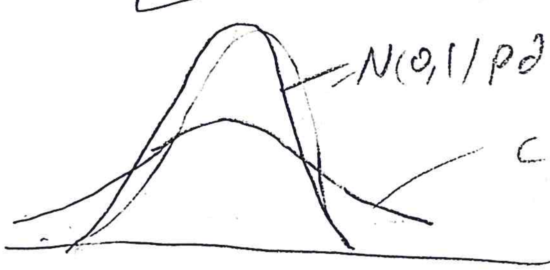
5) Ross uses $c = Np$ = integer.

See 480 n65 n66 for image processing and signal detection, (kernel method, bivariate transformations, correlation)

6) X has a Cauchy(μ, σ) distribution, $X \sim C(\mu, \sigma)$, if the pdf of X is $f(x) = \dfrac{1}{\pi\sigma\left[1 + \left(\frac{y-\mu}{\sigma}\right)^2\right]}$, $y \in \mathbb{R}$.

Then $E(Y)$ and $V(Y)$ do not exist.

$$F(y) = \frac{1}{\pi}\left[\arctan\left(\frac{y-\mu}{\sigma}\right) + \frac{\pi}{2}\right].$$



N(0,1) pdf has "light tails."

Cauchy(0,1) pdf has "heavy tails."

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

20 sheets of notes F Dec 11 2:45-4:45

$\log = \ln$ in this class

1) The key for independence is whether $p(x,y)$ or $f(x,y)$ factors. If the support is not a cross product, $p(x,y)$ and $f(x,y)$ do not factor, so X and Y are dependent. If the support is a cross product, $X \perp\!\!\!\perp Y$

$$\left\{\begin{array}{ll} \text{iff } p(x,y) = p_X(x)\, p_Y(y) & \forall\, x,y \in \text{support} \\ \text{iff } f(x,y) = \underbrace{g(x)\, h(y)}_{\text{easier than } f_X(x) f_Y(y)} & \forall\, x,y \in \text{support}, \end{array}\right.$$

so on cross product support, X and Y are dependent if $p(x,y)$ or $f(x,y)$ do not factor.

ex) (old Exam 2 7d) $f(y_1,y_2) = y_1 + y_2 \neq g(y_1)\, h(y_2)$

so Y_1 and Y_2 are dependent

2) $Y_1 \perp\!\!\!\perp Y_2 \Rightarrow \text{cov}(Y_1, Y_2) = 0$

but $\text{cov}(Y_1, Y_2) = 0$ does not imply $Y_1 \perp\!\!\!\perp Y_2$

ex $Y_2 = Y_1^2$ so dependent but $\text{cov}(Y_1, Y_2) = 0$

"best line" has slope 0

$$\prod_{i=1}^{n} a^{b\theta_i} = a^{\sum_{i=1}^{n} b\theta_i} = a^{b\sum_{i=1}^{n}\theta_i}, \quad \prod_{i=1}^{n} a^{b\theta} = a^{nb\theta}$$

Frev15

$$\prod_{i=1}^{n} e^{b\theta_i} = e^{\sum_{i=1}^{n} b\theta_i} = e^{b\sum_{i=1}^{n}\theta_i}, \quad \prod_{i=1}^{n} e^{b\theta} = e^{nb\theta}$$

$\sum_{i=1}^{n} \theta_i$ can't be reduced, but $\sum_{i=1}^{n} \theta = n\theta$.

ex $Y_1, \ldots, Y_n \overset{ind}{\sim}$ with mgf $\phi_{Y_i}(t) = (pe^t + 1-p)^m$.

Find the mgf of $U = \sum_{i=1}^{n} Y_i$.

soln $\phi_U(t) = \prod_{i=1}^{n} \phi_{Y_i}(t) = \prod_{i=1}^{n} (pe^t + 1-p)^m = [pe^t + 1-p]^{\sum_{i=1}^{n} m}$

$$= \boxed{[pe^t + (1-p)]^{nm}}.$$

ex] $Y_1, \ldots, Y_n$ are ind with mgf $\phi_{Y_i}(t) = \exp[(e^t - 1)\lambda_i]$

The mgf of $U = \sum_{i=1}^{n} Y_i$ is $\phi_U(t) = \prod_{i=1}^{n} \phi_{Y_i}(t) =$

$$\prod_{i=1}^{n} \exp[(e^t - 1)\lambda_i] = \boxed{\exp\left[(e^t - 1)\sum_{i=1}^{n} \lambda_i\right]}$$

ex] $Y_1, \ldots, Y_n$ are ind with mgf $\phi_{Y_i}(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha_i}$.

So the mgf of $U = \sum_{i=1}^{n} Y_i$ is $\prod_{i=1}^{n} \phi_{Y_i}(t) =$

$$\prod_{i=1}^{n} \left(\frac{1}{1-\beta t}\right)^{\alpha_i} = \boxed{\left(\frac{1}{1-\beta t}\right)^{\sum_{i=1}^{n} \alpha_i}}$$

ex) $V(Y_1) = V(Y_2) = \frac{1}{18}$; $Y_1 \perp\!\!\!\perp Y_2$ [illegible]

Find $V(Y_1 - Y_2) = V(Y_1) + V(Y_2) = \frac{1}{18} + \frac{1}{18} = \boxed{\frac{1}{9}}$

ex) $EY = \frac{1}{3}$, $EY^2 = \frac{1}{6}$

Find a) $E[2Y-1] = 2E(Y) - 1 = \frac{2}{3} - 1 = \boxed{-\frac{1}{3}}$

b) $E[1-2Y] = 1 - 2E(Y) = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$

c) $E[Y^2 - 3Y] = \frac{1}{6} - 3\frac{1}{3} = \boxed{-\frac{5}{6}}$

ex)

	y_2 = 0	1	2	$P(Y_1 = y_1)$
y_1 = 0	0/27	2/27	4/27	6/27
1	1/27	3/27	5/27	9/27
2	2/27	4/27	6/27	12/27
$P(Y_2 = y_2)$	$\frac{3}{27}$	$\frac{9}{27}$	$\frac{15}{27}$	

Are Y_1 and Y_2 ind?

No support is not a cross product.

$[\log(y)]^2 = \log(y)\log(y) \neq \log(y^2) = 2\log(y)$

$\prod_{i=1}^{n} a^{r_i} = a^{\sum_{i=1}^{n} r_i}$, $\prod_{i=1}^{n} a^{r} = a^{\sum_{i=1}^{n} r} = a^{nr}$

$10 \cdot 10 \cdot 10 \; - \; 10 \; = 10^{\frac{3}{\cdot}}$ not 10

$\uparrow \qquad = 10^{\frac{3}{2}}$

multiply 10 by itself 6 times

Freud 2.5

---

mgf from pmf

y	-1	3
$p(y)$	$\frac{1}{4}$	$\frac{3}{4}$

pmf from mgf $\phi(t) = \frac{1}{4}e^{-t} + \frac{3}{4}e^{3t}$

y	-1	3
$p(y)$	$\frac{1}{4}$	$\frac{3}{4}$

$$Ee^{tY} = \sum e^{ty} p(y) = \frac{1}{4}e^{-t} + \frac{3}{4}e^{3t} = \phi(t)$$

$$\phi'(t) = -\frac{1}{4}e^{-t} + 3\cdot\frac{3}{4}e^{3t}, \quad \phi'(0) = -\frac{1}{4} + \frac{9}{4} = 2 = \sum y\,p(y) = E(Y)$$

$$\phi''(t) = \frac{1}{4}e^{-t} + \frac{27}{4}e^{3t}, \quad \phi''(0) = \frac{1}{4} + \frac{27}{4} = \frac{28}{4} = 7 = EY^2$$

$$= \sum y^2 p(y) = (-1)^2\frac{1}{4} + 3^2\left(\frac{3}{4}\right)$$

$$V(Y) = EY^2 - (EY)^2 = 7 - (2)^2 = 7 - 4 = 3$$

If pmf $p(y)$ is defined on $-j$ to j, then

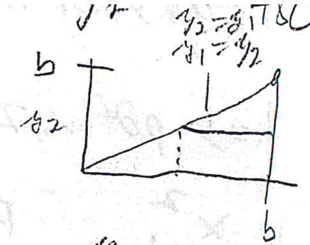
$$\phi(t) = \sum_{y=-j}^{j} e^{ty} p(y)$$

$$\phi'(t) = \sum_{y=-j}^{j} y e^{ty} p(y), \qquad \phi'(0) = \sum_y y\,p(y) = E(Y)$$

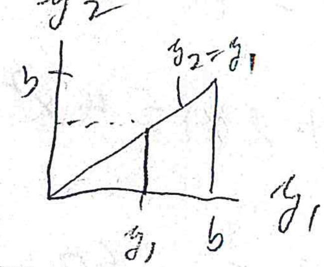
$$\phi''(t) = \sum_{y=-j}^{j} y^2 e^{ty} p(y), \quad \phi''(0) = \sum_y y^2 p(y) = E(Y^2)$$

$$\phi^{(k)}(t) = \sum_{y=-j}^{j} y^k e^{ty} p(y), \quad \phi^{(k)}(0) = \sum_y y^k p(y) = E(Y^k)$$

$0 \le y_2 \le y_1 \le b$

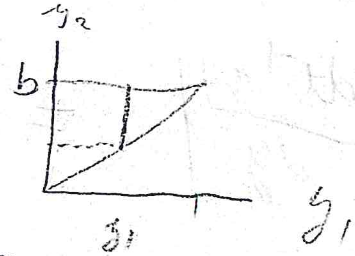


$[y_2, b] =$ "support" of $f_{Y_1|Y_2}(y_1|y_2)$ = limits on dy_1

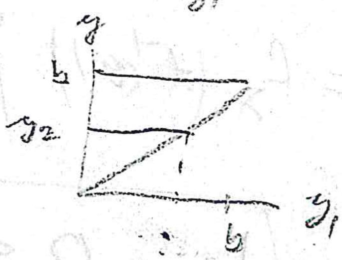


$[0, y_1] =$ "support" of $f_{Y_2}(y_2|y_1)$ = limits on dy_2

$0 \le y_1 \le y_2 \le b$



$[y_1, b] =$ "support" of $f_{Y_2|Y_1}(y_2|y_1)$ = limits on dy_2

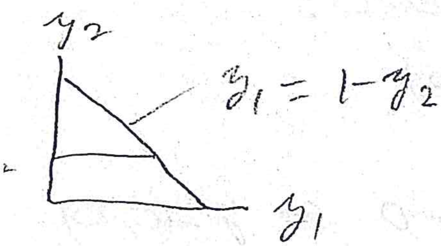


$[0, y_2] =$ "support of $f_{Y_1|Y_2}(y_1, y_2)$ = limits on dy_1

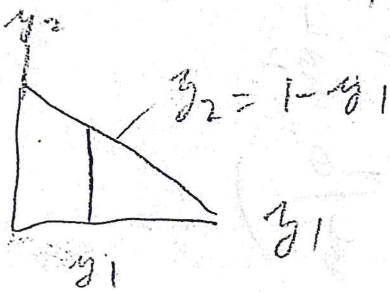
$0 < y_1 = 1$ $0 < y_2 = 1$ $y_1 + y_2 \le 1$

$y_2 = 1 - y_1$

or $y_1 = 1 - y_2$



limits on dy_1 or "support" of $f_{Y_1|Y_2}(y_1|y_2)$ is $0 < y_1 < 1 - y_2$



limits on dy_2 or "support" of $f_{Y_2|Y_1}(y_2|y_1)$ is $0 < y_2 < 1 - y_1$

Freu 3.5

ex] X has pdf $f(x) = 2x$ $\quad 0 < x < 1$.

$Y = 4X^2$. Find the pdf of Y.

soln] $y = t(x) = 4x^2$, $\quad t(0) = 0$, $t(1) = 4$ so $\mathcal{Y} = (0,4)$.

$x^2 = \frac{y}{4}$ so $x = \frac{1}{2} y^{\frac{1}{2}} = t^{-1}(y)$

$$\left| \frac{dt^{-1}(y)}{dy} \right| = \frac{1}{4} y^{-\frac{1}{2}}$$

$$f_Y(y) = f_X(t^{-1}(y)) \left| \frac{dt^{-1}(y)}{dy} \right| = 2 \cdot \frac{1}{2} y^{\frac{1}{2}} \cdot \frac{1}{4} y^{-\frac{1}{2}}$$

$$= \boxed{\frac{1}{4}, \quad 0 < y < 4}$$

ex] X has pdf $f(x) = \frac{\theta}{x^2} \exp\left(-\frac{\theta}{x}\right)$, $x > 0$, $\theta > 0$.

$\mathcal{X} = (0, \infty)$

Find pdf of $Y = \frac{1}{X}$.

soln $y = t(x) = \frac{1}{x}$. $\quad t(0) = \infty$, $t(\infty) = 0$ so $\mathcal{Y} = (0, \infty)$.

$x = \frac{1}{y} = t^{-1}(y) = y^{-1}$, $\quad \left| \frac{dt^{-1}(y)}{dy} \right| = \left| -\frac{1}{y^2} \right| = \frac{1}{y^2}$

$$f_Y(y) = f_X(t^{-1}(y)) \left| \frac{dt^{-1}(y)}{dy} \right| = \frac{\theta}{\left(\frac{1}{y}\right)^2} \exp\left(-\frac{\theta}{\frac{1}{y}}\right) \frac{1}{y^2}$$

$$= \boxed{\theta \exp(-\theta y), \quad y > 0}$$

Common errors Asked for $V(Y_1) = EY_1^2 - EY_1^2$

but give EY_1^2.

7 pages

12 Q's

20 parts

15 pts each

look at Exam 1-3 1st
then Q1-Q11

} more emphasis on later material $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$, $\mu_{\bar{X}} = \mu$

Exam 2 # 2 CLT normal approx for $\bar{X}$

#1 joint pdf → $cov(Y_1, Y_2)$

#7 joint pdf, marginal, EY_1, VY_1 is $Y_1 \perp Y_2$

#8 method of transformations

#4 Poisson process

Exam 3 too

Exam 3 compound Poisson process

nonhomogeneous poisson process

Markov Chain

nonhomogeneous Markov chain

Simulation! $x_i = F^{-1}(U_i)$ give U_1, U_2

arithmetic Brownian motion

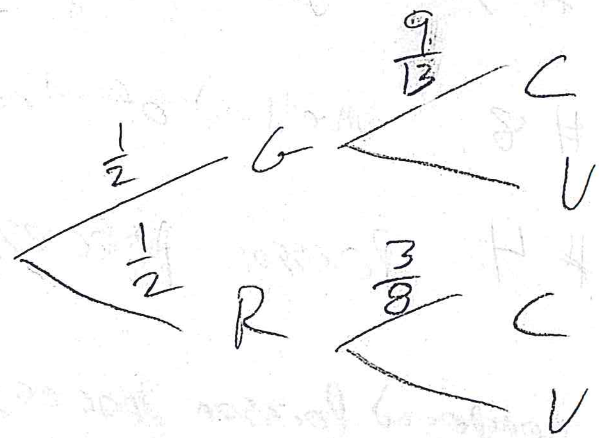
Exam 1 ≈ marginal stuff EY_1, EY_1^2, VY_1 from $f(y_1, y_2)$

Bayes Th

ex) A red jar contains 5 vanilla and 3 chocolate cookies. A green jar contains 4 vanilla and 9 chocolate cookies. A jar is grabbed at random and one cookie selected at random. If the cookie selected is chocolate, what is the prob that the green jar was selected?

Soln want $P(G|C) = \dfrac{P(G)\,P(C|G)}{P(G)\,P(C|G) + P(\bar{G})\,P(C|\bar{G})}$

$$= \frac{P(G \wedge C)}{P(G \wedge C) + P(\bar{G} \wedge C)}$$



$$P(G|C) = \frac{\frac{1}{2}\,\frac{9}{13}}{\frac{1}{2}\,\frac{9}{13} + \frac{1}{2}\,\frac{3}{8}} = \frac{.3462}{.53371} = \boxed{.6487}$$

Frev4-5