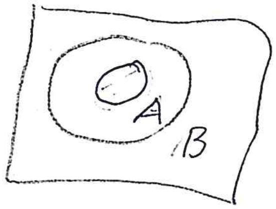
sum to one   A conditional "prob

$P(\cdot/M)$ is a prob with sample space M

36} If $A \subset B$, then $P(A \wedge B) = P(A)$
since $A \wedge B = A$.



§1.5

p9

37} * Two events A and B are <u>independent</u> if $P(AB) = P(A)P(B)$
or if any <u>one</u> of the following 3 conditions holds

I1) $P(A \wedge B) = P(A)\,P(B)$

I2) $P(A|B) = P(A)$ (if $P(B) > 0$

I3) $P(B|A) = P(B)$ if $P(A) > 0$)

Otherwise, A and B are <u>dependent</u>.

38} Interpretation: If $P(A)$ is unaffected by the knowledge that B occured or did not occur then A and B are independent.

Fact} If A and B are ind, then so are $\bar{A}$ and B, $\bar{A}$ and $\bar{B}$, and A and $\bar{B}$.

y A 100% HW grade B = it rained today in England
A and B are ind

39) <u>common EI problem</u>! Given some of the following probs $P(A)$, $P(B)$, $P(A \cap B)$ and $P(A|B)$, find $P(A|B)$ and whether A and B are independent.

ex) $P(A) = 0.4$, $P(B) = 0.3$, $P(AB) = 0.12$.

So $P(A|B) = \frac{P(AB)}{P(B)} = \frac{.12}{.3} = 0.4 = P(A)$.

$P(AB) = .12 = .4(.3) = P(A)P(B)$ > So A and B are ind

40) * Suppose $P(A) > 0$ and $P(B) > 0$. If A and B are mutually exclusive, then knowing A occurred means B did not occur. Being disjoint is an extreme form of dependence.

$P(AB) = 0$, $P(A|B) = 0$

41) <u>Common EI Problem</u>! Given a table, determine if a row event and column event are ind.

ex)

	A	B	C	D	F	rowtot
M	11	8	3	1	2	25
Other	19	41	40	1	37	138
coltot	30	49	43	2	39	163

Are A and M ind? $P(A \cap M) = \frac{11}{163} \neq \frac{30}{163} \cdot \frac{25}{163} = P(A)P(M)$. not ind

42) mult rules

$P(A \cap B) = P(A)\,P(B|A) = P(B)\,P(A|B)$

<u>mult rule for 2 ind. events</u> $P(A \cap B) = P(A)\,P B$

<u>mult rule for ind. events</u>: If $A_1, A_2, \dots, A_k$ are ind,

$$P(A_1 \cap A_2 \cap \dots \cap A_k) = P(A_1)\,P(A_2) \cdots P(A_k) = \underbrace{\prod_{i=1}^{k}}_{\text{multiply}} P(A_i)$$

← most general

$$P(A_1 \cap A_2 \cap \dots \cap A_k) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \dots \cap A_{k-1})$$

proof: use def of cond prob and everything cancels

43) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ gen add rule

$P(A \cup B) = P(A) + P(B)$ if A and B are disjoint

$P(A \cup B) = P(A) + P(B) - P(A)P(B)$ if A and B are ind

44) <u>Common Ex problem</u>: given 3 of the 4 prob's in the gen add rule, find the 4th.

variant: You are told A and B are ind
or A and B are disjoint.

ex) $P(A \cup B) = .8$, $P(A) = .7$, $P(B) = .6$

) Find $P(A \cap B)$

) Are A and B disjoint?

Are A and B ind?

$= P(A) + P(B) - P(A \cup B)$

) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ so $P(A \cap B) = .7 + .6 - .8 = .5$

$P(A) + P(B) = .7 + .6 > 1$ so not disjoint also $P(A \cap B) = .5 \neq 0$

$P(A \cap B) = .5 \neq .7(.6) = P(A)P(B)$ so not ind

$P(A) = .4$ $P(B) = 0.$

95

a) Find $P(A \cup B)$ if A and B are disjoint,

b) " " independent,

Soln a) $P(A \cup B) = P(A) + P(B) = .4 + .3 = 0.7$

b) $P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{P(A)P(B)} = .4 + .3 - (.4)(.3) = 0.58$

§1.6

45) $B_1, \ldots, B_k$ ..... partition S if

$\underbrace{B_i \cap B_j = \phi \text{ for } i \neq j}_{\text{pairwise disjoint}}$, $\underbrace{P(B_i) > 0}_{B_i \neq \phi}$ for $i = 1, \ldots, k$, and $S = \bigcup_{i=1}^{k} B_i = B_1 \cup \cdots \cup B_k$.

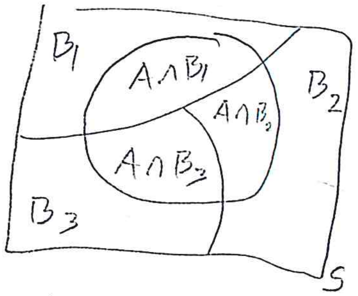
Let $A \subseteq S$. Then $A = \underbrace{(A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_k)}_{\text{disjoint}}$

Law of total probability: $P(A) = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(A|B_i) P(B_i)$.

This law is useful when $P(A|B_i)$ and $P(B_i)$ are given for $i = 1, \ldots, k$.

know for E1

46) Bayes' rule: Let $B_1, \ldots, B_k$ partition S.



the circle is A

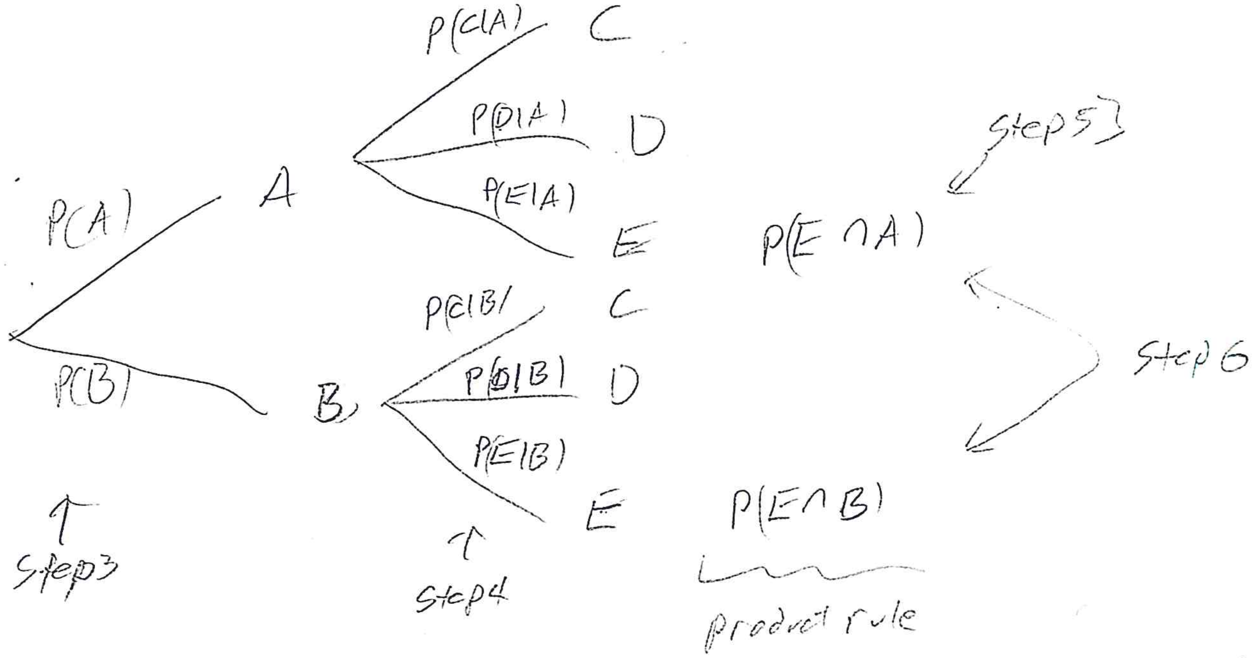
Then $$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j) P(B_j)}{\sum_{i=1}^{k} P(A|B_i) P(B_i)}.$$

If $k=2$, $$P(A|E) = \frac{P(A)\, P(E|A)}{P(A)\, P(E|A) + P(\bar{A})\, P(E|\bar{A})}.$$

47) Problem diagrams are useful for solving a Bayes rule problem.

step 1) Figure out what conditional prob you want eg $P(A|E)$.

step 2) draw tree diagram eg A, B partition S



step 3) The left branches have the unconditional probs.

step 4) The right branches have p(right branch | left branch).

step 5) The product of the right branch probability with a the corresponding left branch probability is p(left branch ∩ right branch), so find

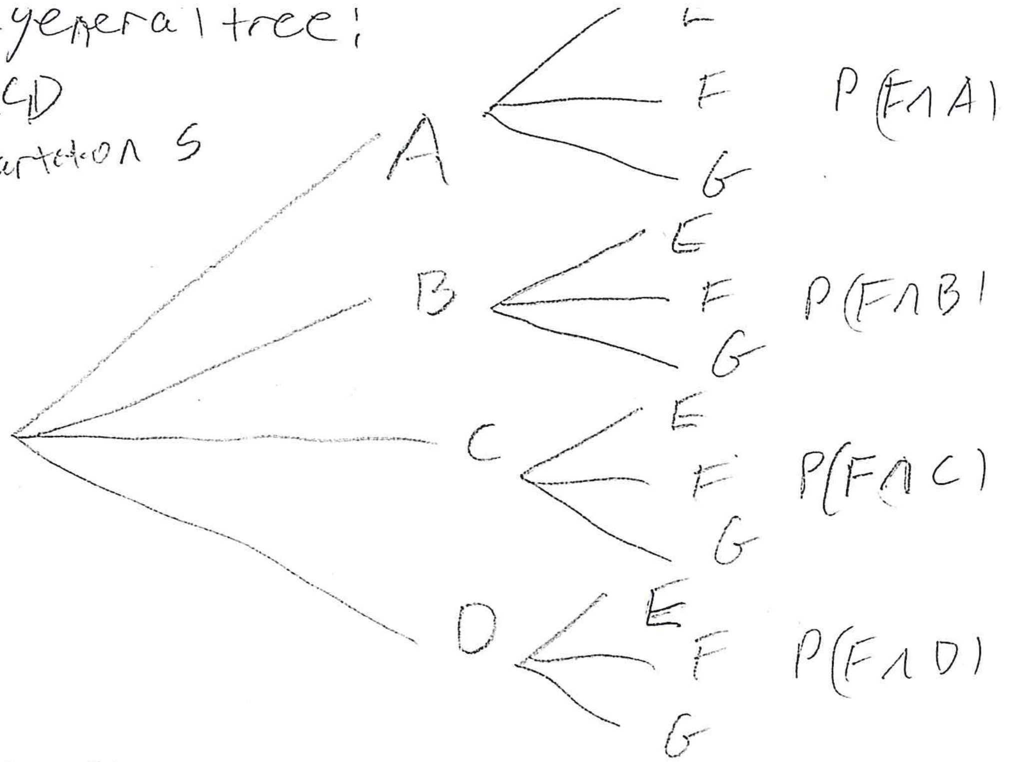
$$P(E \cap A) = P(A \cap E) = P(A)\,P(E|A).$$

step 6) $P(A|E) = \dfrac{P(A \cap E)}{P(E)}$, $A \cup B = S,\ A \cap B = \emptyset$

so $P(E) = P(E \cap A) + P(E \cap B)$.

step 7) $P(A|E) = \dfrac{P(E \cap A)}{P(E \cap A) + P(E \cap B)} = \dfrac{P(A \cap E)}{P(A \cap E) + P(A \cap B)}$ ← numerator is always in denominator more general tree:

A, B, C, D partition S



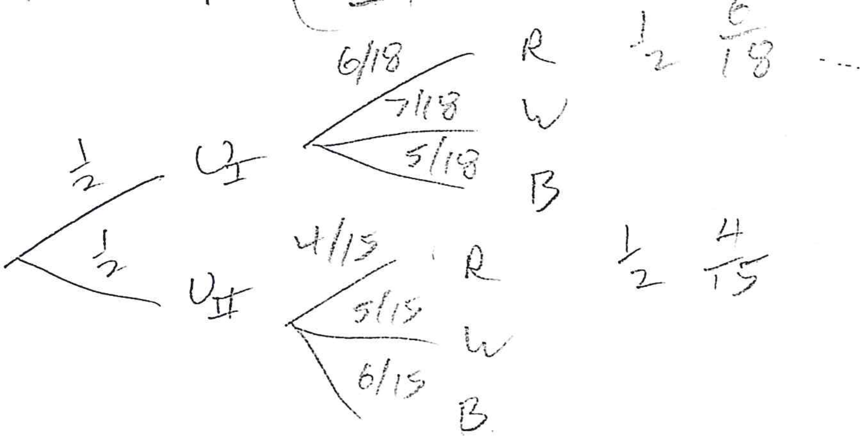
$$P(B|F) = \frac{P(B\cap F)}{P(F)} = \frac{P(F\cap B)}{P(F\cap A) + P(F\cap B) + P(F\cap C) + P(F\cap D)}$$

num term is alway a denom term

ex} Urn I has 6R, 7W, 5Blue marbles.
urn II has 4R, 5W, 6B marbles.

a) If a marble selected is red, what is the prob it came from Urn II?

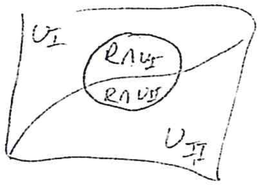
soln} want $P(\overset{\text{left}}{U_{II}} / \overset{\text{right}}{R})$



$$P(U_{II} \mid R) = \frac{P(U_{II} \cap R)}{P(R)} = \frac{P(U_{II} \cap R)}{P(U_I \cap R) + P(U_{II} \cap R)}$$

$$= \frac{\frac{1}{2}\,\frac{4}{15}}{\frac{1}{2}\,\frac{4}{15} + \frac{1}{2}\,\underbrace{\frac{6}{18}}_{\frac{1}{3}}} = \frac{\frac{4}{15}}{\frac{4}{15} + \underbrace{\frac{5}{15}}_{\frac{1}{3}}} = \boxed{\frac{4}{9} = 0.4444}$$

b) What is the prob that the marble is red?



$$P(R) = P(R \cap U_I) + P(R \cap U_{II})$$

$$= \frac{1}{2}\,\frac{6}{18} + \frac{1}{2}\,\frac{4}{15} = \frac{1}{6} + \frac{2}{15} = \frac{5+4}{30} = \frac{9}{30} = \frac{3}{10} = \boxed{0.3}$$

c) what is $P(B \mid U_{II})$?

$$= \frac{6}{15} = \frac{\#\text{ blue in Urn II}}{\#\text{ marbles in Urn II}}$$ from tree

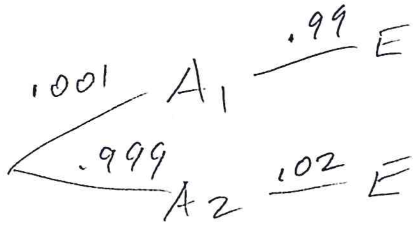
ex) There are many tests for rare diseases and a positive result means that the medical test suggests (perhaps incorrectly) that the person has the disease. Suppose the test is such that if the person has the disease, then a positive result occurs 99% of the time. Suppose a person without the disease tests positive 2% of the time. Assume $\frac{1}{1000}$ people screened have the disease. If a randomly

selected (screened) [illegible] what is ?
the prob the person has the disease? 11.5

soln} Let A_1 denote the event that the randomly selected person has the disease and $A_2 = \overline{A_1}$ (the event that the person does not have the disease).
If E is the event that the test gives a positive result, want $P(A_1 | E)$

$$= \frac{P(E|A_1)\,P(A_1)}{P(E|A_1)\,P(A_1) + P(E|A_2)\,P(A_2)} = \frac{.99(.001)}{.99(.001) + .02(.999)}$$

≈ 0.047. Instead of telling the person she has the rare disease, the doctor should inform her that she is in a high risk group and needs more testing.



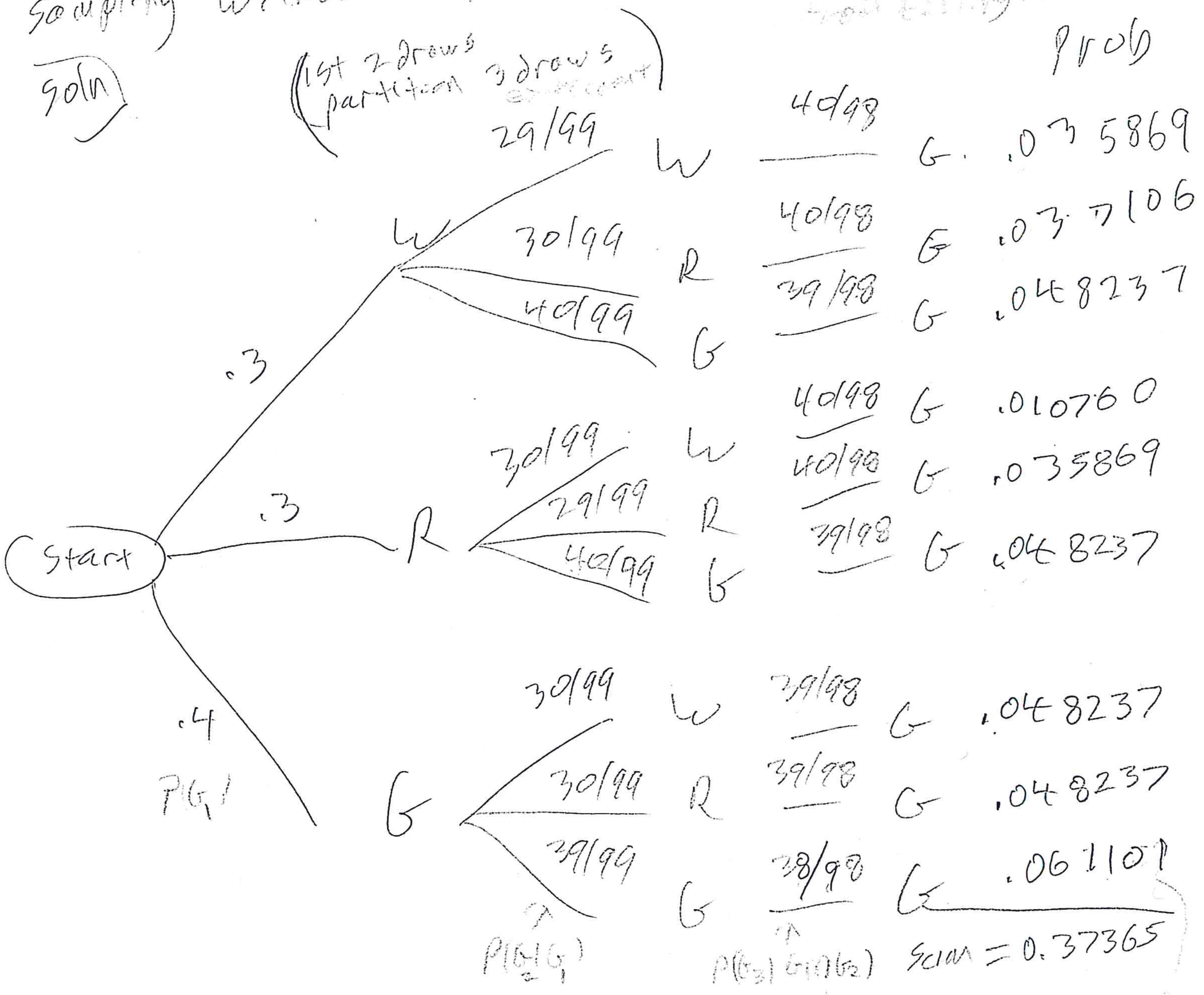
48). small trees can be useful to solve complicated probability problems, but as the number of branches increases, it is more likely that a mistake will be made.

ex) 100 balls : 30W , 30R , 40G .

~~skip~~ Skip ex
Find prob 3rd ball drawn is G when sampling without replacement .

or give tree and filling in probs

soln



multiply prob's along each branch and sum

↑ skip

$$P(G_1 \cap G_2 \cap \dots)$$

ex) Find $P(G_1 G_2 G_3) = .4 \; \frac{39}{99} \; \frac{38}{98} = 0.061101$

$P(G_1) \; P(G_2|G_1) \; P(G_3|G_1 \cap G_2) = P(G_1 \, G_2 \, G_3)$

12.5

marbles. Draw a marble from each urn, what is the prob that one marble is W and one is G?

Soln} $P(\underset{\uparrow \text{1st}}{W_1}\underset{\uparrow \text{2nd}}{G_2}) + P(\underset{\uparrow \text{1st}}{G_1}\underset{\uparrow \text{2nd}}{W_2}) \overset{\text{the urns are ind}}{=} \frac{2}{10}\frac{3}{10} + \frac{3}{10}\frac{2}{10} = \boxed{\frac{12}{100}}$

can omit subscripts if $WG = W_1G_1$, $GW = G_1W_2$,

Not $P(W \text{ and } G) = P(W \wedge G) = P(W)P(G) = \frac{2}{10}\frac{3}{10} = \frac{6}{100}$

because "event" W is not a well defined set in the 2 draw experiment (unless W is 1st draw G 2nd draw), P(W) needs WC!

(W_1G_2 and G_1W_2 are events so are $\underset{\substack{\uparrow \\ \text{W on 1st draw}}}{W_1}$, W_2, G_1 and $\underset{\substack{\uparrow \\ \text{G on 2nd draw}}}{G_2}$)

49}^{p10} Events $E_1, \dots, E_n$ are independent if every subset $E_{i1}, \dots, E_{ir}$ of these events, $r \le n$ satisfies $P(E_{i1} \wedge E_{i2} \wedge \dots \wedge E_{ir}) = \prod_{j=1}^{r} P(E_{ij}) = P(E_{i1}) \dots P(E_{ir})$

50}^{p10} E_1, E_2, E_3 can be pairwise ind

$P(E_i \wedge E_j) = P(E_i)P(E_j)$ for $i \neq j$, but

E_1, E_2, and E_3 are dependent.

ex} $S = \{1, 2, 3, 4\}$ equally likely $E_1 = \{1, 2\}$, $E_2 = \{1, 3\}$, $E_3 = \{1, 4\}$

$P(E_i \wedge E_j) = \frac{1}{4} = P(E_i)P(E_j) = \frac{1}{2}\frac{1}{2}$, but

$P(E_1E_2E_3) = P(\{1\}) = \frac{1}{4} \neq P(E_1)P(E_2)P(E_3)$