$p_{Y_2}(0) = P(\text{eng}) = P(0,0) + P(1,0) = \frac{?}{26} + \frac{?}{26} = \frac{?}{26}$ (29)

$P_{Y_2}(1) = P(\text{non-eng}) = P(0,1) + P(1,1) = \frac{5}{26} + \frac{9}{26} = \frac{14}{26}$

Note: The marginal pmf of Y_i is just the univariate pmf of Y_i.

k	0	1
$p(Y_1 = k)$	$\frac{8}{26}$	$\frac{18}{26}$

$\Leftarrow P_{Y_1}(y) \in [0,1]$ and $\sum P_{Y_1}(y) = 1$

E2 rev p.5

80) Know p43 If Y_1 and Y_2 have joint pdf $f(y_1, y_2)$, then the marginal pdfs of Y_1 and Y_2 are

don't match

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 \quad \forall\, y_1$$

hold fixed

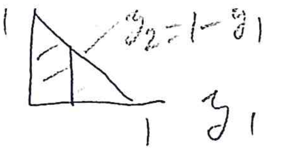
$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1, \quad \forall\, y_2.$$

don't match

Note: A marginal pdf is the univariate pdf.

81) Common Problem! Find a marginal pdf from a joint pdf or a marginal pmf from a joint pmf. See last ex.

ex) $f(y_1, y_2) = \begin{cases} 24\, y_1 y_2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1,\ y_1 + y_2 \le 1 \\ 0 & \text{else} \end{cases}$



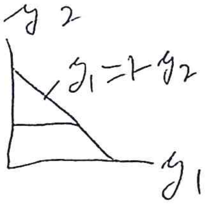
Make [illegible]

what you are integrating! $dy_2 \parallel y_2$ axis (24.5

$$f_{Y_1}(y_1) = \overbrace{\int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 = \int_0^{1-y_1}}^{\text{do match}} 24 y_1 y_2\, dy_2 \quad \text{don't match}$$

$$= 24 y_1 \frac{y_2^2}{2}\Big|_0^{1-y_1} = 12 y_1 (1-y_1)^2, \quad 0 \le y_1 \le 1$$

By symmetry, $f_{Y_2}(y_2) = 12 y_2 (1-y_2)^2, \quad 0 \le y_2 \le 1$



$$\text{or } f_{Y_2}(y_2) = \overbrace{\int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 = \int_0^{1-y_2}}^{\text{do match}} 24 y_1 y_2\, dy_1 \quad \text{don't match}$$

$$= 24 \frac{y_1^2}{2} y_2 \Big|_0^{1-y_2} = 12 (1-y_2)^2 y_2, \quad \underbrace{0 \le y_2 \le 1}_{\text{support is crucial}}$$

p45

32) Know RVs Y_1 and Y_2 are independent (written $Y_1 \perp\!\!\!\perp Y_2$)

if any one of the following conditions holds:

i) $F(y_1, y_2) = F_{Y_1}(y_1) F_{Y_2}(y_2) \quad \forall\, y_1, y_2 \in \mathbb{R}$

ii) $p(y_1, y_2) = p_{Y_1}(y_1) p_{Y_2}(y_2) \quad \forall\, y_1, y_2 \in \mathbb{R}$, Y_1, Y_2 discrete

iii) $f(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2) \quad \forall\, y_1, y_2 \in \mathbb{R}$, Y_1, Y_2 contin

otherwise Y_1, Y_2 are dependent.

33) Know $Y_1, \ldots, Y_n$ are ind if

$F(y_1, \ldots, y_n) = F_{Y_1}(y_1) F_{Y_2}(y_2) \cdots F_{Y_n}(y_n)$

or if $p(y_1, \ldots, y_n) = p_{Y_1}(y_1) \cdots p_{Y_n}(y_n) \quad Y_i$ discrete or if $F(y_1, \ldots, y_n) = F_{Y_1}(y_1) \cdots F(y_n)$ $\qquad$ Y_i contin 30

$\forall y_1, \ldots, y_n \in \mathbb{R}$.

Otherwise $Y_1, \ldots, Y_n$ are dependent.

84) Known
Let Y_1 and Y_2 have a joint pdf $f(y_1, y_2)$ that is positive iff (if and only if)

support of Y_1 and Y_2 $\left\{ \begin{array}{l} a \leq y_1 \leq b \text{ and } c \leq y_2 \leq d \text{ for} \\ \text{constants } a, b, c, d \text{ (possibly } \pm \infty). \end{array} \right.$

Then Y_1 and Y_2 are independent iff

$f(y_1, y_2) = g(y_1)\, h(y_2)$ on the support

where g is a nonnegative function of y_1 alone

and h $\qquad$ y_2

85) The support $\mathcal{Y}_i$ of $Y_i = \{y : f_{Y_i}(y) > 0 \text{ or } p_{Y_i}(y) > 0\}$.

The support $\mathcal{Y}$ of Y_1 and Y_2 =

$\{(y_1, y_2) : f(y_1, y_2) > 0 \text{ or } p(y_1, y_2) > 0\}$.

The support $\mathcal{Y}$ is a cross product if

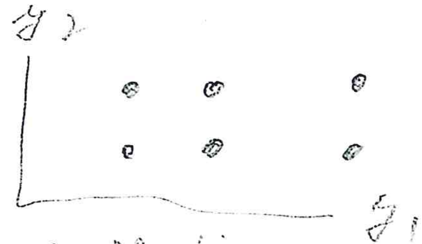
$\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 = \{(y_1, y_2) : y_1 \in \mathcal{Y}_1 \text{ and } y_2 \in \mathcal{Y}_2\}$.

ex) The support is rectangular if

$\mathcal{Y}_1$ and $\mathcal{Y}_2$ are intervals eg $\{(y_1, y_2) : y_1 \in (a, \infty), y_2 \in (c, d)\}$.

86) * A necessary (but not sufficient) condition

... 2 discrete RVs Y_1 and Y_2 to be ind is that $p(y_1, y_2)$ is positive on a grid that is a cross product.

ex 

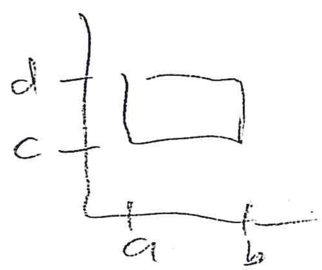
Support of Y_1 and Y_2

support is a cross product
Y_1 and Y_2 could be ind or dep
check $p(y_1, y_2) = p_{Y_1}(y_1) p_{Y_2}(y_2)$ on the grid, if so, $Y_1 \perp\!\!\!\perp Y_2 \xleftarrow{} $ ind

Y_1 and Y_2 are dependent: support is not a cross product
$p(3,3) = 0 \neq p_{Y_1}(3) p_{Y_2}(3) > 0$
Also if $Y_1 = 4$ then $Y_2 = 1$.

If $p(y_1, y_2) \neq p_{Y_1}(y_1) p_{Y_2}(y_2)$ for some entry (y_1, y_2) on the support, then Y_1, Y_2 are dep

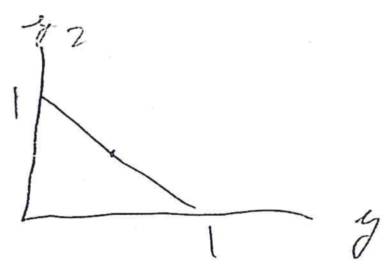
87) * A necessary condition for 2 contin RVs to be ind is that the support of Y_1 and Y_2 is a cross product



could be ind or dep

check $f(y_1, y_2) = g(y_1) h(y_2)$ on support

Support of Y_1 and Y_2



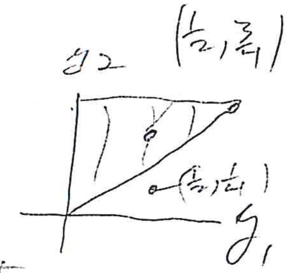
Y_1 and Y_2 are dependent: support is not a cross product.

$f(\frac{1}{2}, \frac{3}{2}) = 0 \neq f_{Y_1}(\frac{1}{2}) f_{Y_2}(\frac{3}{2}) > 0$

Also if Y_1 is near 1 then Y_2 is near 0.

...) $f(y_1,y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{else} \end{cases}$

Let $g(y_1) = 6$ for $0 \le y_1 \le 1$ and $h(y_2) = 1-y_2$ for $0 \le y_2 \le 1$. Are Y_1 and Y_2 ind?



Soln} No, support is not a cross product.

Also $f(\frac{1}{2},\frac{1}{4}) = 0 \ne f_{Y_1}(\frac{1}{2})\, f_{Y_2}(\frac{1}{4}) > 0$.

88} A necessary condition for $Y_1,\ldots,Y_n$ to be ind is that the support $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$ is a cross product (cartesian product).

89} Suppose $Y_1,\ldots,Y_n$ are ind and $\mathcal{Y}_i$ is an interval with endpoints a_i, b_i with $a_i = -\infty$ $b_i = \infty$ possible.

Then $E\left[\prod_{i=1}^{n} g_i(Y_i)\right] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} g_i(y_i)\, f(y_1,\ldots,y_n)\, dy_1 \cdots dy_n$

$= \prod_{i=1}^{n} \int_{a_i}^{b_i} g_i(y_i) \underbrace{f_{Y_i}(y_i)}_{\text{marginal}} dy_i = \prod_{i=1}^{n} E[g_i(Y_i)]$

if the integrals exist. Similar result for pmfs.

90, If $p(y_1,y_2)$ is given by a table, $Y_1 \perp\!\!\!\perp Y_2$ if (ith row sum)(jth col sum) $= p_{Y_1}(y_1)\, p_{Y_2}(y_2) = p(y_1,y_2) =$ ij th table entry for all ij entries.

ex

		1	—	—	1	
Y_1	1	.06	.02	.04	.08	.2
	2	.15	.05	.10	.20	.5
	3	.09	.03	.06	.12	.3
		.3	.1	.2	.4	

31.5

$Y_1 \perp\!\!\!\perp Y_2$ since all 12 products check!

$.3(.2)$ $.1(.2)$ $.2(.2)$ $.4(.2)$
$.3(.5)$ $.1(.5)$ $.2(.5)$ $.4(.5)$
$.3(.3)$ $.1(.3)$ $.2(.3)$ $.4(.3)$

Note: dependence is easier; stop as soon as one pair (y_1, y_2) is found such that $p(y_1, y_2) \neq p_{Y_1}(y_1)\, p_{Y_2}(y_2)$,

91) If $p(y_1, \ldots, y_n)$ is a joint pmf, then

$$E\, g(Y_1, \ldots, Y_n) = \sum_{y_n} \sum_{y_{n-1}} \cdots \sum_{y_1} g(y_1, \ldots, y_n)\, p(y_1, \ldots, y_n).$$

92) If $f(y_1, \ldots, y_n)$ is a joint pdf, then

$$E\, g(Y_1, \ldots, Y_n) = \int_{\chi_n} \int_{\chi_{n-1}} \cdots \int_{\chi_1} g(y_1, \ldots, y_n)\, f(y_1, \ldots, y_n)\, dy_1 \cdots dy_n$$

where the χ_i are the limits of integration for dy_i.

93) Usually $n = 2$ and $g(y_1, y_2) = y_1^i\, y_2^j$ for small integers i and j. $E(Y_1)$, $E(Y_2)$, $E(Y_1 Y_2)$ are common

94) Know $E[g(Y_i)] = \int_{-\infty}^{\infty} g(y_i) \underbrace{f_{Y_i}(y_i)}_{\text{marginal}}\, dy_i$

as before.

95) Hard way! $E[g(Y_1)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1) f(y_1, y_2)\, dy_2\, dy_1$

$= \int_{-\infty}^{\infty} g(y_1) \underbrace{\left[\int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2\right]} dy_1$

$= \int_{-\infty}^{\infty} g(y_1)\, f_{Y_1}(y_1)\, dy_1$

96) easy way! $E[g(Y_i)] = \sum_{y_i} g(y_i) \underbrace{P_{Y_i}(y_i)}_{\text{marginal}}$

hard way $E[g(Y_1)] = \sum_{y_1}\sum_{y_2} g(y_1) P(y_1, y_2)$

$= \sum_{y_1}\left[g(y_1) \sum_{y_2} P(y_1, y_2)\right] = \sum_{y_1} g(y_1) P_{Y_1}(y_1)$

97) Given $P(y_1, y_2)$ or $f(y_1, y_2)$

$E(Y_1) = \sum_{y_1} y_1 P_{Y_1}(y_1)$ or $\int_{-\infty}^{\infty} y\, f_{Y_1}(y)\, dy$

$E(Y_2) = \sum_{y_2} y_2 P_{Y_2}(y_2)$ or $\int_{-\infty}^{\infty} y\, f_{Y_2}(y)\, dy$

$E\, g(Y_1) = \sum_{y_1} g(y_1) P_{Y_1}(y_1)$ or $\int_{-\infty}^{\infty} g(y) f_{Y_1}(y)\, dy$

$E\, g(Y_2) = \sum_{y_2} g(y_2) P_{Y_2}(y_2)$ or $\int_{-\infty}^{\infty} g(y) f_{Y_2}(y)\, dy$

98) $E(C) = C$ for any constant

$E\left[\sum_{i=1}^{n} g_i(Y_1, Y_2)\right] = \sum_{i=1}^{n} E[g_i(Y_1, Y_2)]$

In particular $E(aY_1+bY_2) = aE(Y_1) + bE(Y_2)$ 32.5

99} * p46 Let $Y_1 \perp\!\!\!\perp Y_2$, and $g(Y_1)$ and $h(Y_2)$ functions of only Y_1 and Y_2, respectively. Then

$$\underbrace{E[g(Y_1)h(Y_2)]}_{\text{joint}} = \underbrace{E[g(Y_1)]\, E[h(Y_2)]}_{\text{univariate}}$$

provided the expectations exist.

In particular $E[Y_1 Y_2] = E(Y_1)E(Y_2)$ if $Y_1 \perp\!\!\!\perp Y_2$. see 85}

Proof for discrete case} $E[g(Y_1)h(Y_2)] =$

$$\sum_{y_1}\sum_{y_2} g(y_1)h(y_2)\,p(y_1,y_2) \overset{\text{ind}}{=} \sum_{y_1}\sum_{y_2} g(y_1)h(y_2)\,p_{Y_1}(y_1)\,p_{Y_2}(y_2)$$

$$= \sum_{y_1} g(y_1)\,p_{Y_1}(y_1) \sum_{y_2} h(y_2)\,p_{Y_2}(y_2) = E[g(Y_1)]\, E[h(Y_2)].$$

100} p46 know The covariance of Y_1 and Y_2 is

$$\text{cov}(Y_1, Y_2) = E[(Y_1 - E(Y_1))(Y_2 - E(Y_2))].$$

101} p47 know Short cut formula

$$\text{cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2).$$

102} * $\text{cov}(Y_i, Y_i) = V(Y_i)$ for $i = 1, 2$.