$$V\left[S_N \mid N\right] = V\left[\sum X_i\right] = \sum V(X_i)$$

N is a constant given N $\quad X_i|N \sim X_i$ since $X_i \perp N$ (conditional dist = marginal if $X \perp N$)

So $V(S_N) = E\left[V(S_N|N)\right] + V\left[E(S_N|N)\right] =$

$$E\left[N\,V(X_i)\right] + V\left[N\,E(X)\right] = E(N)\,V(X_i) + V(N)\left[E(X)\right]^2$$

$\square$

14] E2 problem: Given "$Y|X \sim W$" and $X \sim U$ find $E(Y)$ and $V(Y)$. ($Y|X=x$ is a RV, $Y|X$ is a family of RVs)

ex] $Y \mid P = p \sim bin(n, p)$, $P \sim beta(\alpha=4, \beta=6)$.

Find a) $E(Y)$ b) $V(Y)$.

Soln] a) $E(Y) = E\left[\underbrace{E(Y|P)}_{bin(n,P)}\right] = E(nP) = n\,E(P) = n\,\frac{\alpha}{\alpha+\beta}$

$$= n\,\frac{4}{4+6} = \boxed{0.4\,n}$$

b) $V(Y) = E\left[V(\underbrace{Y|P}_{bin(n,P)})\right] + V\left[E(\underbrace{Y|P}_{bin(n,P)})\right] =$

$$E\left[nP(1-P)\right] + V\left[nP\right] = n\,E(P) - n\,\underbrace{E(P^2)}_{V(P)+\left[E(P)\right]^2} + n^2\,V(P)$$

$$= n\,\frac{\alpha}{\alpha+\beta} - n\left[\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \left(\frac{\alpha}{\alpha+\beta}\right)^2\right] + n^2\,\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$= \boxed{0.021816\,n^2 + 0.21818\,n}$

455

ex) $Y|X \sim bin(X, P)$ and $X \sim poisson(\lambda)$
$\uparrow$ RV

Then $E[Y|X] = XP$ and $V(Y|X) = XP(1-P)$

$E(X) = V(X) = \lambda$,

So $E(Y) = E\left[E[Y|X]\right] = E(XP) = P\,E(X) = \boxed{P\lambda}$

$V(Y) = E\left[V(Y|X)\right] + V\left[E(Y|X)\right] =$

$E\left[X\,P(1-P)\right] + V(XP) = \lambda P(1-P) + P^2 V(X)$

$= \lambda P - \lambda P^2 + P^2\lambda = \boxed{\lambda P}$

§3.5 153 P115 Let $X = \begin{cases} 1 & E \text{ occurs} \\ 0 & E \text{ doesn't occur.} \end{cases}$

The indicator RV $X \sim bin(n=1, P=P(E))$

Then $E(X) = P(E)$, $E[X|Y=y] = P(E|Y=y)$,

$P(E) = \sum_y P(E|Y=y)P(Y=y)$ or $P(E) = \int_{-\infty}^{\infty} P(E|Y=y) f_Y(y)\,dy$

ex) If $X \perp\!\!\!\perp Y$ have pdf's, $P(X<Y) = \dots = \int_{-\infty}^{\infty} F_X(y) f_Y(y)\,dy$.

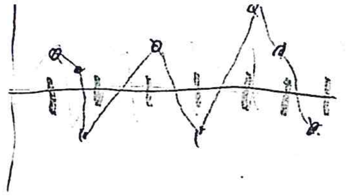
End probability = M483 material, Skip § 3.6-3.7.
Actuarial exam P! Final 2018 (from syllabus) is good. Ross is ok for probability but good for stochastic process.

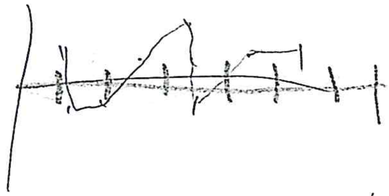
1) p77 A <u>stochastic process</u> $\{X(t); t \in T\}$ is a collection of random variables.
Hence for each $t \in \underbrace{T}_{\text{tau}}$, $X(t)$ is a random variable.
Often t is time and $X(t)$ is the state of the process at time t. Often $T = (0, \infty)$, $[0, \infty)$ or $\{1, 2, 3, \ldots\}$.

2) Often the stochastic process has been observed at $t_1 < t_2 < \cdots < t_n$, and the random variables $X(t_1), \ldots, X(t_n)$ are dependent, Sometimes you can find the distribution of $W = X(t_0)$ or $W = X(t_2) - X(t_1)$.
Then you can find $E(W)$, $V(W)$, $P(W \le x)$ etc,

3) A stochastic process $\{X(t); t \in \{1, 2, 3, \ldots\}\}$ is a <u>white noise</u> if $X_1, \ldots, X_t, \ldots$ are iid with $E(X_i) = 0$ and $V(X_i) = \sigma^2$ and $X_t = X(t)$.

4) The plot of t_i vs $X(t_i)$ is a <u>sample path</u>.





4b-5

5} p78 discrete time stochastic process T is countable

continuous time T is an interval

p185, 198, 152, 424

6) A stochastic process $\{Y(t) : t \in \{1,2,3,\ldots\}\}$ is a <u>random walk</u> if $Y(t) = Y_t = Y_{t-1} + e_t$ where the e_t are iid and $Y_0 = y_0$ is a constant.

Then $Y_t = Y_{t-2} + e_{t-1} + e_t = Y_{t-3} + e_{t-2} + e_{t-1} + e_t = \cdots$

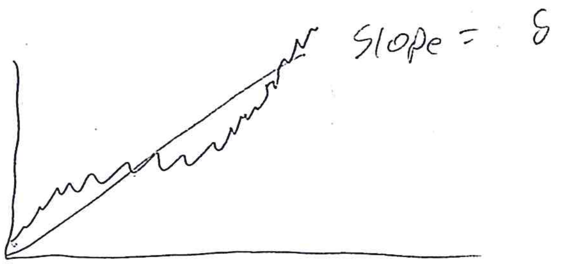
$= Y_{t-j} + e_{t-j+1} + \cdots + e_t = \cdots = y_0 + e_1 + \cdots + e_t$

$= y_0 + \sum_{i=1}^{t} e_i$ where $\sum_{i=1}^{t} e_i$ is known as

a <u>cumulative sum</u>. If $E(e_i) \equiv \delta$ and $V(e_i) \equiv \sigma^2$,

then $E\,Y_t = E\left[y_0 + \sum_{i=1}^{t} e_i\right] = y_0 + t\delta$ and

$V(Y_t) = V\left(y_0 + \sum_{i=1}^{t} e_i\right) \overset{iid}{=} \sum_{i=1}^{t} V(e_i) = t\sigma^2$



sample path 1

random walk

Sample path 2

Assume $P(e_i = 0) \neq 1$
so $P(e_i = 0) < 1$.

See HW 8

7) E2 rev 38) gave $\sum_{i=1}^{n} e_i$ where $e_i = x_i$ for 8) brand name RVs.

ex) If $y_0 = 0$ and $e_t \sim EXP(\lambda)$, then

$$X(t) = \sum_{i=1}^{t} e_i \sim G(t, \lambda), \quad t \in 1, 2, \ldots.$$

8) The random walk model is often used for stock prices. (drunkard's walk, Random Walk Down Wall Street: monkey throwing darts at list of stocks can have similar performance to that of managers of stock funds).

ch 5

9) p297 A stochastic process $\{N(t), t \geq 0\}$ is a <u>counting process</u> if $N(t)$ counts the total number of events that occurred in a time interval $(0, t]$.

0) p297 For any fixed $t_0 > 0$, $N(t_0)$ is a RV and counts the number of events that occurred in $(0, t_0]$.

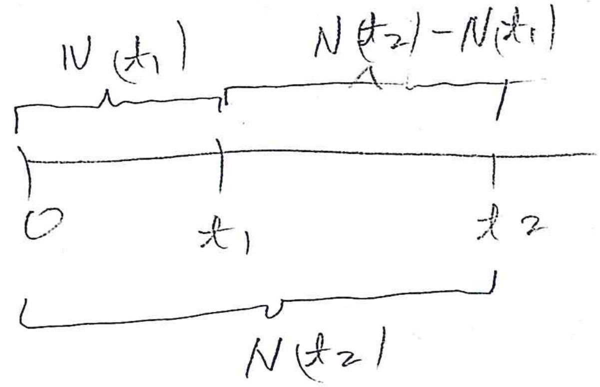
1) Properties of $N(t)$:

1) $N(t) \geq 0$.

ii) $N(t)$ is integer ...

iii) If $0 < t_1 < t_2$, $N(t_1) \le N(t_2)$.

iv) If $0 < t_1 < t_2$, $N(t_2) - N(t_1)$ = number of events that occurred in interval $(t_1, t_2]$.



12} p297 $N(t)$ is said to possess <u>independent increments</u> if the number of events that occur in <u>disjoint</u> time intervals are independent.

So $N(t_2) - N(t_1)$ and $N(t_4) - N(t_3)$ are independent if $(t_1, t_2]$ and $(t_3, t_4]$ are disjoint.

Note that $(t_1, t_2]$ and $(t_2, t_3]$ are disjoint if $t_1 < t_2 < t_3$. (Assume $t_1 < t_2 < \dots < t_n < \dots$)

13} p298 $N(t)$ is said to have <u>stationary increments</u> if the distribution of the number of events that occur in any time interval depends only on the length of the interval: $(t_1, t_2]$ has length $t_2 - t_1$.

14) * The counting process $\{N(t), t \geq 0\}$ is a Poisson process having rate λ for $\lambda > 0$ if

i) $N(0) = 0$.
ii) The process has independent increments.
iii) The number of events in any interval of length t has a Poisson(λt) distribution with mean λt.

15] Hence the poisson process has stationary increments.

16] Consider the intervals $0 \leq t \leq t_0$

$$W = N(t_0) \sim \text{Poisson}(\lambda t_0).$$

$$E[N(t_0)] = V[N(t_0)] = E(W) = V(W) = \lambda t_0 = \mu.$$

$$P(N(t_0) = k) = P(W = k) = \frac{e^{-\lambda t_0}(\lambda t_0)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

17] Consider the intervals $s \leq t^* \leq s + t$

variable time   fixed

$$W = D(t) = N(t+s) - N(s) \sim \text{Poisson}(\lambda t).$$

$$E[N(t+s) - N(s)] = V(N(t+s) - N(s)) = E(W) = V(W) = \lambda t$$

$$P(N(t+s) - N(s) = k) = P(W = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad k = 0, 1, \ldots$$

18} Consider interval $t_1 \le t \le t_2$, with $t_1 < t_2$.

$W = N(t_2) - N(t_1) \sim \text{Poisson}[\lambda(t_2 - t_1)]$. 48s

$E(W) = V(W) = E(N(t_2) - N(t_1)) = V(N(t_2) - N(t_1)) = \lambda(t_2 - t_1) = \mu$

$$P(W = k) = P(N(t_2) - N(t_1) = k) = \frac{e^{-\lambda(t_2 - t_1)} [\lambda(t_2 - t_1)]^k}{k!}$$

for $k = 0, 1, 2, \ldots$

Let $\mu = \lambda(t_2 - t_1)$, $E(W) = V(W) = \mu$, $P(W = k) = \frac{e^{-\mu}\mu^k}{k!}$, $k = 0, 1, \ldots$

ex} For traffic and phone calls, often the busiest hour $\simeq$ Poisson.

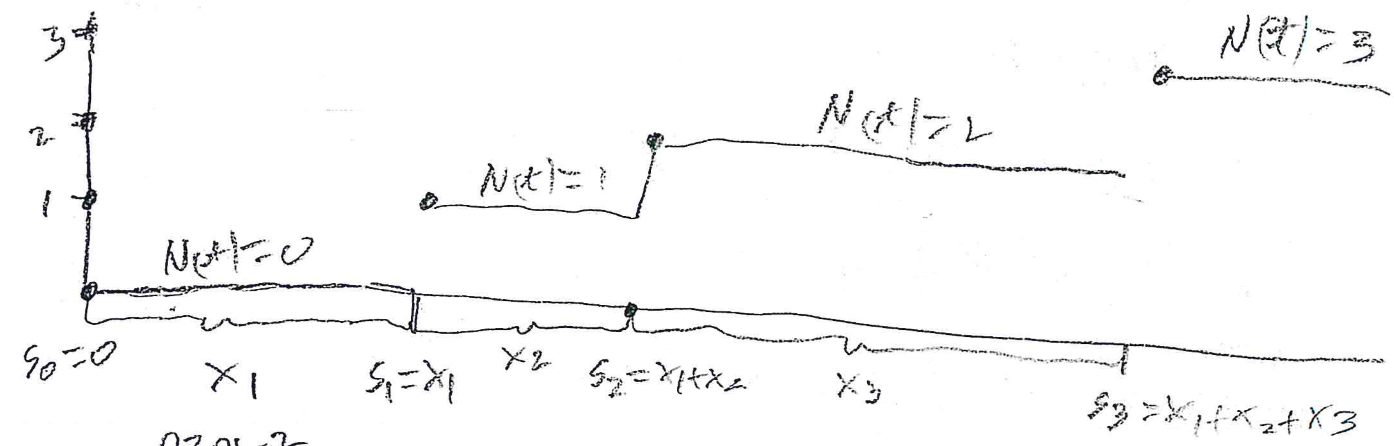
19} $N(0) = 0$ means # of events in $(t, t+\Delta]$ = # of events in $(t, t+\Delta)$.
[ )

20} p301 Let X_1 be the waiting time from time $= 0$ until the 1st event. Let X_2 be the waiting time after the 1st event until the 2nd event. Let X_j be the waiting time after the $(j-1)$th event until the jth event.

Then $S_n = \sum_{i=1}^{n} X_i$ = time of occurrence of nth event

= waiting time until the nth event.

... see HW 8.
paths of a Poisson process



p301-2

21} Know For a poisson process with rate λ, the X_i are iid EXP(λ) with $E(X_i)=1/\lambda$. Hence $S_n = S_{n-1}+X_n$ is a random walk with $S_0=0$.

$S_n \sim \text{Gamma}(n,\lambda)$, $E(S_n)=\frac{n}{\lambda}$, $V(S_n)=\frac{n}{\lambda^2}$.

22} Know Assume that if the waiting times = interarrival times are iid EXP(λ), then $N(t)$ is a Poisson Process with rate λ.

(21) and 22) are equivalent under a mild regularity conditions. Billingsley Th 23.1 p310.)

ex} Organ transplant claims have interarrival times that are iid EXP with a mean of $\frac{1}{4}$ ($\frac{1}{4}$th of a month) So $\frac{1}{\lambda}=\frac{1}{4}$ and $N(t)$ is a poisson process with rate 4 per month where $N(t)$ counts number of claims. So $X_i \sim \text{EXP}(\lambda=4)$.

ex} If $N(t)$ is a poisson process with $\lambda=60$ per hour, then p311 events should occur every $\left(\frac{1}{60}\right)$th hour on ave, (160 / 49.5)

23) know Suppose $N(t)$ is a Poisson process with rate λ that counts events of k distinct types where $p_i = P(\text{type } i \text{ event})$. If $N_i(t)$ counts events of type i, then $N_i(t)$ is a poisson process with rate $\lambda_i = \lambda p_i$ and the $N_i(t)$ are ind for $i = 1, \ldots, k$.

Then $N(t) = \sum_{i=1}^{k} N_i(t)$, $\lambda = \sum_{i=1}^{k} \lambda_i$ and $\sum_{i=1}^{k} p_i = 1$.

24) p322 know The counting process $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function or rate function $\lambda(t)$, also called a nonstationary Poisson process, with the following properties.

i) $N(0) = 0$

ii) the process has independent increments

iii) $N(t)$ is a Poisson$(m(t))$ RV where $m(t) = \int_0^t \lambda(r)\,dr$ and $N(t)$ counts the number of events in $(0, t]$ (or $(0,t)$).

counts the number of events in $(t_1, t_2]$ where

$$m(t_2) - m(t_1) = \int_{t_1}^{t_2} \lambda(r)\, dr.$$

pois process $\lambda(t) \equiv \lambda$ $m(t) = \lambda t$

25) The nonstationary Poisson process does not have stationary increments, but $N(t+s) - N(s) \sim \text{Poisson}(m(t+s) - m(s))$ where

$$m(t+s) - m(s) = \int_s^{t+s} \lambda(r)\, dr.$$

26) Let $N(t)$ be a Poisson process with rate λ. Suppose there are k distinct events where the prob of the ith event occuring at time s is $P_i(s)$. Let $N_i(t)$ count type i events.

so the prob of the ith event depends on time

Then $N_i(t)$ is a nonhomogeneous Poisson process with $\lambda_i(t) = \lambda \int_0^t P_i(s)\, ds$.

Here $\sum_{i=1}^{k} P_i(s) = 1$ and the $N_i(t)$ are ind RVs for fixed t.

27) p327 Know A stochastic process $\{X(t), t \ge 0\}$ is a compound Poisson process if $X(t) = \sum_{i=1}^{N(t)} Y_i$, $t \ge 0$ where $\{N(t), t \ge 0\}$ is a Poisson process with

rate λ; and $\{Y_n : n \geq 0\}$ is a family of iid RVs independent of $N(t)$. (eg Y_i = ith insurance claim)

$X(t) = 0$ if $N(t) = 0$. Then $E[X(t)] = \lambda t\, E(Y_1)$ 50.5

and $V(X(t)) = \lambda t\, E[Y_i^2]$. $X(0) = 0$.

ex) If $Y_1 \equiv 1$, $X(t) = N(t)$ is the usual Poisson process.

$X(t) = \underbrace{1 + \cdots + 1}_{N(t)} = N(t)$

28) The compound Poisson process has ind and stationary increments. Fix $r, t > 0$. Then

${}_tX_r = X(r+t) - X(r)$ has the same dist

as the RV $X(t)$. Thus $E[{}_tX_r] = \lambda t\, E(Y_1)$

and $V[{}_tX_r] = \lambda t\, E[Y_i^2]$.

29) The parameters of $X(t)$ are

λt, $\underbrace{E(Y_1) \text{ and } E(Y_1^2)}_{\text{or the distribution of } Y \text{ eg } F_Y(y)}$

30) In financial mathematics (actuarial), often the Y_i are loss severity RVs ($\underbrace{\text{loss in dollars}}_{\text{insurance claim}}$) while $N(t)$ counts the number of losses over $(0, t]$. Then $X(t) = \sum_{i=1}^{N(t)} Y_i = Y_1 + Y_2 + \cdots + Y_{N(t)}$ is