$Y_1, \ldots, Y_n$ iid MLE practice *

1) $N(\mu,1)$: $\log L(\mu) = C - \frac{1}{2}\sum(y_i-\mu)^2$, $\frac{d}{d\mu}\log L(\mu) = \sum Y_i - n\mu$, $\hat{\mu} = \overline{Y}$

$\frac{d^2}{d\mu^2}\log L(\mu) = -n$ www .edu/olive/ichp.pdf

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2) $\mathrm{Exp}(\beta)$: $\log L(\beta) = -n\log\beta - \frac{\sum Y_i}{\beta}$, $\frac{d}{d\beta}\log L(\beta) = -n\frac{1}{\beta} + \frac{\sum Y_i}{\beta^2}$, $\hat{\beta} = \overline{Y}$

$$\frac{d^2}{d\beta^2}\log L(\beta) = \frac{n}{\beta^2} - \frac{2\sum Y_i}{\beta^3}\Big|_{\beta=\hat{\beta}} = \frac{-n}{(\overline{Y})^2} < 0$$

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3) $\mathrm{bin}(n=1,p)$: $\log L(p) = C + \sum y_i \log p + (n - \sum Y_i)\log(1-p)$

$\frac{d}{dp}\log L(p) = \sum Y_i \frac{1}{p} - (n - \sum Y_i)\frac{1}{1-p}$, $\hat{p} = \overline{Y}$

$$\frac{d^2}{dp^2}\log L(p) = -\sum Y_i \frac{1}{p^2} - (n - \sum Y_i)\frac{1}{(1-p)^2}\Big|_{p=\overline{Y}} = \frac{-n}{(\overline{Y})} - \frac{n}{(1-\overline{Y})} < 0$$

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4) $\mathrm{poisson}(\lambda)$ $\log L(\lambda) = C + \sum y_i \log(\lambda) - n\lambda$

$\frac{d}{d\lambda}\log L(\lambda) = \sum y_i \frac{1}{\lambda} - n$, $\hat{\lambda} = \overline{Y}$,

$$\frac{d^2}{d\lambda^2}\log L(\lambda) = \frac{-\sum y_i}{\lambda^2}\Big|_{\lambda=\overline{Y}} = -\frac{n}{\overline{Y}} < 0 \text{ if } \overline{Y} \neq 0$$

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5) $\mathrm{Geometric}(p)$: $\log L(p) = n\log(p) + (\sum Y_i - n)\log(1-p)$

$\frac{d}{dp}\log L(p) = n\frac{1}{p} - \frac{\sum Y_i - n}{1-p}$, $\hat{p} = (\overline{Y})^{-1}$

$$\frac{d^2}{dp^2}\log L(p) = \frac{-n}{p^2} - \frac{\sum Y_i - n}{(1-p)^2} < 0$$ since $Y_i \geq 1$ so $\sum Y_i \geq n$

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6) $N(\mu,\sigma^2)$, μ known $\log L(\sigma^2) = C - \frac{n}{2}\log\sigma^2 - \frac{\sum(y_i-\mu)^2}{2\sigma^2}$

$\frac{d}{d\sigma^2}\log L(\sigma^2) = \frac{-n}{2}\frac{1}{\sigma^2} + \frac{\sum(Y_i-\mu)^2}{2\sigma^4}$, $\hat{\sigma}^2 = \frac{1}{n}\sum(Y_i-\mu)^2$

$$\frac{d^2}{d(\sigma^2)^2}\log L(\sigma^2) = \frac{n}{2\sigma^4} - \frac{\sum(Y_i-\mu)^2}{\sigma^6}\Big|_{\sigma^2=\hat{\sigma}^2} = \frac{-n}{2(\hat{\sigma}^2)^2} < 0$$

MLE* a)

1) $Y_1, \ldots, Y_n$ iid $N(\mu, \sigma^2)$, $\sigma^2 > 0$ known

$$L(\mu) = \prod \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(y_i - \mu)^2\right] = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2}$$

$$\log L(\mu) = \log\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n - \frac{1}{2\sigma^2}\sum(y_i-\mu)^2$$

$$\frac{d\log L(\mu)}{d\mu} = -\frac{1}{2\sigma^2}\, 2\sum(y_i-\mu)(-1) = \frac{1}{\sigma^2}\sum(y_i-\mu)$$

$$= \frac{1}{\sigma^2}\left(\sum Y_i - n\mu\right) \overset{\text{set}}{=} 0$$

or $\sum Y_i = n\mu$, $\hat{\mu} = \overline{Y}$ unique

$$\frac{d^2 \log L(\mu)}{d\mu^2} = -\frac{n}{\sigma^2} < 0$$

$\overline{Y}$ is MLE

2) $Y_1, \ldots, Y_n$ iid $\exp(\beta)$

$$L(\beta) = \prod \frac{1}{\beta} e^{-y_i/\beta} = \left(\frac{1}{\beta}\right)^n e^{-\sum y_i/\beta}$$

$$\log L(\beta) = -n\log\beta - \frac{1}{\beta}\sum y_i$$

$$\frac{d\log L(\beta)}{d\beta} = -n\frac{1}{\beta} + \frac{1}{\beta^2}\sum y_i \overset{\text{set}}{=} 0$$

$\sum Y_i = n\beta$, $\hat{\beta} = \overline{Y}$ unique

$\overline{Y}$ is MLE

$$\frac{d^2\log L(\beta)}{d\beta^2} = \frac{n}{\beta^2} - \frac{2\sum Y_i}{\beta^3}\Bigg|_{\beta=\overline{Y}} = \frac{n}{(\overline{Y})^2} - \frac{2n\overline{Y}}{(\overline{Y})^3} = \frac{-n}{(\overline{Y})^2} < 0$$

MLE b)

3) bin$(n=1, p)$ $\quad L(p) = \prod_i p^{y_i} (1-p)^{1-y_i}$

$\binom{1}{0} = \binom{1}{1} = 1$ $\quad = p^{\Sigma y_i} (1-p)^{n-\Sigma y_i}$

$$\log L(p) = \Sigma y_i \log p + (n - \Sigma y_i) \log(1-p)$$

$$\frac{d \log L(p)}{dp} = \Sigma y_i \frac{1}{p} - (n - \Sigma y_i) \frac{1}{(1-p)} \overset{set}{=} 0$$

$$(1-p) \Sigma y_i = p (n - \Sigma y_i)$$

$$\Sigma y_i = p \Sigma y_i + pn - p \Sigma y_i = pn$$

$$\hat{p} = \bar{y}, \text{ unique}$$

$$\frac{d^2 \log L(p)}{dp^2} = -\frac{\Sigma y_i}{p^2} - \frac{(n - \Sigma y_i)}{(1-p)^2} < 0 \text{ if } \Sigma y_i \neq 0$$

since $y_i \in \{0, 1\}$
so $0 \le \Sigma y_i \le n$

$$\text{or } \left. \frac{d^2 \log L(p)}{dp^2} \right|_{p=\bar{y}} = -\frac{n}{\bar{y}} - \frac{n}{(1-\bar{y})} < 0$$

since $n - \Sigma y_i = n(1-\bar{y})$

$y_1, \dots, y_n$ iid with $p(y) = \binom{m}{y} p^y (1-p)^{m-y}$

bin(m, p): $L(p) = \prod_i \binom{m}{y_i} p^{y_i} (1-p)^{m-y_i} = \left[\prod_i \binom{m}{y_i}\right] \left[p^{\Sigma y_i}\right] (1-p)^{nm - \Sigma y_i}$

$$\log L(p) = c + \Sigma y_i \log(p) + (nm - \Sigma y_i) \log(1-p)$$

$$\frac{d}{dp} \log L(p) = \Sigma y_i \frac{1}{p} - (nm - \Sigma y_i) \frac{1}{1-p} \overset{set}{=} 0$$

$$(1-p) \Sigma y_i = p (nm - \Sigma y_i) \text{ or } \Sigma y_i = nmp; \quad \hat{p} = \frac{\Sigma y_i}{nm} = \frac{1}{m} \bar{y}$$

$$\frac{d^2}{dp^2} \log L(p) = -\Sigma y_i \frac{1}{p^2} - (nm - \Sigma y_i) \frac{1}{(1-p)^2} < 0$$

since $y_i \in \{0, \dots, m\}$
so $nm \ge \Sigma y_i$

unique

MLE c) *

4) poisson(λ): $L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} = \frac{\lambda^{\Sigma y_i} e^{-n\lambda}}{\prod y_i!}$.

$L(\lambda) = \prod_{i=1}^{n} p(y_i)$

$\log L(\lambda) = \log\left(\frac{1}{\prod y_i!}\right) + \Sigma y_i \log\lambda - n\lambda$

$\frac{d \log L(\lambda)}{d\lambda} = \Sigma y_i \frac{1}{\lambda} - n \overset{set}{=} 0$

or $n\lambda = \Sigma y_i$, $\hat{\lambda} = \bar{Y}$. unique

$\frac{d^2}{d\lambda^2} \log L(\lambda) = -\frac{\Sigma y_i}{\lambda^2}\Big|_{\lambda=\bar{Y}} = \frac{-n\bar{Y}}{(\bar{Y})^2} = \frac{-n}{\bar{Y}} < 0$

if $\bar{Y} \neq 0$ so $\bar{Y}$ is the MLE of λ

5) geometric(p): $L(p) = \prod_{i=1}^{n} p(y_i) = \prod_{i=1}^{n} p(1-p)^{(y_i-1)} = p^n (1-p)^{\Sigma y_i - n}$

$\log L(p) = n\log(p) + (\Sigma y_i - n)\log(1-p)$

$\frac{d \log L(p)}{dp} = n\frac{1}{p} - (\Sigma y_i - n)\frac{1}{(1-p)} \overset{set}{=} 0$

or $n(1-p) = p(\Sigma y_i - n)$ or $n = np + p\Sigma Y_i - np$

or $n = p\Sigma Y_i$ so $\hat{p} = \frac{1}{\bar{Y}}$ unique

$\frac{d^2 \log L(p)}{dp^2} = -\frac{n}{p^2} - \frac{(\Sigma y_i - n)}{(1-p)^2} < 0$ since $y_i \geq 1 \Rightarrow \Sigma Y_i \geq n$

so $\hat{p} = \frac{1}{\bar{Y}}$ is the MLE of p.

6) $N(\mu, \sigma^2)$ $\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(y-\mu)^2}$, μ known MLE d)

using $\gamma = \sigma^2$ can be useful.

$$L(\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(y_i-\mu)^2}$$

$$\log L(\sigma^2) = c - \frac{n}{2}\log\sigma^2 - \frac{\sum(y_i-\mu)^2}{2\sigma^2}, \quad c = \log\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}$$

$$\frac{d \log L(\sigma^2)}{d\sigma^2} = \frac{-n}{2}\frac{1}{\sigma^2} + \frac{\sum(y_i-\mu)^2}{2\sigma^4} \overset{set}{=} 0$$

$$\text{or} \quad \frac{n}{2}\sigma^2 = \frac{\sum(y_i-\mu)^2}{2}$$

$$\text{or} \quad \hat{\sigma}^2 = \frac{\sum(y_i-\mu)^2}{n}, \ \text{unique}$$

$$\frac{d^2 \log L(\sigma^2)}{d(\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{\sum(y_i-\mu)^2}{\sigma^6}\Bigg|_{\sigma^2=\hat{\sigma}^2}$$

$$= \frac{n}{2(\hat{\sigma}^2)^2} - \frac{n\,\frac{\sum(y_i-\mu)^2}{n}}{(\hat{\sigma}^2)^3}$$

$$= \frac{n}{2(\hat{\sigma}^2)^2} - \frac{n}{(\hat{\sigma}^2)^2} = \frac{-n}{2(\hat{\sigma}^2)^2} < 0$$

ex] $f(y) = \frac{2\gamma^{3/2}}{\sqrt{\pi}} y^2 e^{-\gamma y^2}$, $y \in \mathbb{R}$, $\gamma > 0$. MLE e)

Spiegel

a) Find the MLE of γ.

b) Find the MLE of γ^2.

Soln a) $L(\gamma) = \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \frac{2}{\sqrt{\pi}} \gamma^{3/2} y_i^2 e^{-\gamma y_i^2}$

$$= \left(\frac{2}{\sqrt{\pi}}\right)^n \left(\prod_{i=1}^{n} y_i^2\right) \gamma^{\frac{3n}{2}} e^{-\gamma \sum_{i=1}^{n} y_i^2}$$

$$\log L(\gamma) = \log\left[\left(\frac{2}{\sqrt{\pi}}\right)^n \prod y_i^2\right] + \frac{3n}{2}\log\gamma - \gamma\sum_{i=1}^{n} y_i^2$$

$$\frac{d \log L(\gamma)}{d\gamma} = \frac{3n}{2}\frac{1}{\gamma} - \sum_{i=1}^{n} y_i^2 \stackrel{\text{set}}{=} 0$$

or $\frac{1}{\gamma} = \frac{2}{3n}\sum_{i=1}^{n} y_i^2$ or $\hat{\gamma} = \frac{3}{2}\frac{n}{\sum_{i=1}^{n} y_i^2}$ unique

Now $$\frac{d^2 \log L(\gamma)}{d\gamma^2} = -\frac{3n}{2}\frac{1}{\gamma^2} < 0$$

So $\hat{\gamma} = \frac{3}{2}\frac{n}{\sum_{i=1}^{n} y_i^2}$ is the MLE of γ.

b) $(\hat{\gamma})^2 = \left(\frac{3}{2}\frac{n}{\sum y_i^2}\right)^2$ is the MLE of γ^2

by invariance

ex] $f(y) = r\,(2-2y)\,(2y-y^2)^{r-1}$, $0<y<1$, MLE f) $r>0$.

a) Find the MLE of r.

b) Find the MLE of $\frac{1}{r}$.

Topp Leone

a) $L(r) = \prod_i f(y_i) = r^n \prod_i (2-2y_i) \prod_i (2y_i - y_i^2)^{r-1}$

$\log L(r) = \log\left(\prod (2-2y_i)\right) + n\log(r) + (r-1)\sum_{i=1}^{n} \log(2y_i - y_i^2)$

$$\frac{d\log L(r)}{dr} = \frac{n}{r} + \sum_{i=1}^{n} \log(2y_i - y_i^2) \overset{set}{=} 0$$

or $n + r\sum_{i=1}^{n}\log(2y_i - y_i^2) = 0$

or $\hat{r} = \dfrac{-n}{\sum_{i=1}^{n}\log(2y_i - y_i^2)}$, unique ✓

$$\frac{d^2\log L(r)}{dr^2} = -\frac{n}{r^2} < 0 \rightarrow \text{so } \hat{r} = \frac{-n}{\sum_{i=1}^{n}\log(2Y_i - Y_i^2)}$$

is the MLE of r.

b) $\dfrac{1}{\hat{r}} = \dfrac{\sum_{i=1}^{n}\log(2Y_i - Y_i^2)}{-n}$ is the MLE of $\frac{1}{r}$

by invariance