15 parts 20 each

Math 483 Final Fall 2015 Name____________________

1) Let $Y_1, ..., Y_n$ be a random sample from a distribution with pdf

$U(Y) = \tau$

$$f(y) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\tau}\right)^{\frac{1}{2}} \exp\left(\frac{-(y^2)}{2\tau}\right)$$

$N(0, \tau)$

4 got

35/40

where $\tau > 0$ and y is real.

a) Find the maximum likelihood estimator of τ. (Make sure that you prove that your answer is the MLE.)

$$L(\tau) = \prod_{i=1}^{n} f(y_i) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\tau}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2\tau}\sum y_i^2\right]$$

$$\log L(\tau) = \log\left(\frac{1}{\sqrt{2\pi}}\right)^n - \frac{n}{2}\log\tau - \frac{1}{2\tau}\sum y_i^2$$

$$\frac{d\log L(\tau)}{d\tau} = -\frac{n}{2}\frac{1}{\tau} + \frac{1}{2\tau^2}\sum Y_i^2 \overset{set}{=} 0$$

or $\sum Y_i^2 = n\tau$ or $\boxed{\hat{\tau} = \frac{\sum_{i=1}^{n} Y_i^2}{n}}$, unique

at least -13 if wrong method or τ

$$\frac{d^2\log L(\tau)}{d\tau^2} = \frac{n}{2}\frac{1}{\tau^2} - \frac{\sum Y_i^2}{\tau^3}\Big/_{\hat{\tau}} =$$

$$\frac{n}{2\hat{\tau}^2} - \frac{n\hat{\tau}}{\hat{\tau}^3} = \frac{n}{2\hat{\tau}^2} - \frac{2n}{2\hat{\tau}^2} = -\frac{n}{2\hat{\tau}^2}\left(= \frac{-n^3}{2(\sum Y_i^2)^2}\right) < 0$$

b) What is the maximum likelihood estimator of $\sqrt{\tau}$? Explain.

~~$\log(\hat{\tau}) = \log\left(\frac{\sum Y_i^2}{n}\right)$~~ by invariance

$$\sqrt{\hat{\tau}} = \sqrt{\frac{\sum_{i=1}^{n} Y_i^2}{n}}$$

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$f_{Y_2}(y_2) = \int_0^1 6y_1^2 y_2 \, dy_1 = 6\frac{y_1^3}{3} y_2 \Big|_{y_1=0}^{1} = 2y_2, \; 0<y_2<1$

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2) Suppose that the joint pdf of the random variables Y_1 and Y_2 is given by

$$f(y_1, y_2) = 6y_1^2 y_2$$

if $0 < y_1 < 1$ and $0 < y_2 < 1$, and $f(y_1, y_2) = 0$, otherwise.

a) Find the marginal pdf of Y_1. Include the support.

$\int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 = \int_0^1 6 y_1^2 y_2 \, dy_2 = 6 y_1^2 \frac{y_2^2}{2}\Big|_0^1$

$= \boxed{3 y_1^2, \quad 0 < y_1 < 1}$

480
Q6d18

or →

b) Find $E(Y_1)$. $= \int y_1 f(y_1)\, dy_1 = \int_0^1 3y_1^3 \, dy_1 = 3\frac{y_1^4}{4}\Big|_0^1$

$= \boxed{\frac{3}{4} = 0.75}$

c) Find $V(Y_1)$. $EY_1^2 = \int y_1^2 f(y_1)\, dy_1 = \int_0^1 3y_1^4 \, dy_1 = \frac{3y_1^5}{5}\Big|_0^1 = \frac{3}{5}$

$V(Y_1) = E(Y_1^2) - (EY_1)^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \boxed{\frac{3}{80} = 0.0375}$

d) Are Y_1 and Y_2 independent? Explain.

Yes, $f(y_1, y_2) = g(y_1)h(y_2)$ on cross product support

or

$= \underbrace{3y_1^2 \, (2y_2)}_{f_{Y_1}(y_1)\, f_{Y_2}(y_2)} = (6y_1^2)\, y_2$ } hard way

80

3). Let $Y_1, \ldots, Y_n$ be a random sample from a distribution with pdf

$$f(y) = \frac{2}{\theta^2}(\theta - y) \quad \text{where} \quad 0 < y < \theta$$

or -15

and $\theta > 0$.

a) Find E(Y). $= \int_0^\theta y \frac{2}{\theta^2}(\theta - y)\,dy = \int_0^\theta \frac{2}{\theta} y\,dy - \int_0^\theta \frac{2}{\theta^2} y^2\,dy$

$$= \frac{2}{\theta}\frac{y^2}{2}\Big|_0^\theta - \frac{2}{\theta^2}\frac{y^3}{3}\Big|_0^\theta = \frac{\theta^2}{\theta} - \frac{2}{\theta^2}\frac{\theta^3}{3} = \theta - \frac{2}{3}\theta = \boxed{\frac{\theta}{3}}$$

b) Find the method of moments estimator for θ.

$$\frac{\theta}{3} \overset{set}{=} \overline{Y} \quad \text{or} \quad \boxed{\hat{\theta} = 3\overline{Y}}$$

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4) Suppose that the number of home fires started by candles is approximately normal and that the appropriate confidence interval can be used. From data from 7 recent years, the mean number of fires was $\overline{x} = 7041.40$ and the standard deviation of of the number of fires was $s = 1610.30$. Find a 99% confidence interval for the mean number of fires started by candles.

omitted but right of -6

$df = n-1 = 6$

6	3.707
	99%

$$\overline{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}} = 7041.4 \pm 3.707\,\frac{1610.3}{\sqrt{7}}$$

$$= 7041.4 \pm 2256.2143$$

$$= \boxed{[4785.1857,\ 9297.6143]}$$

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-14 2.576 (6284.05, 7798.74)

$k\, y^{k-1}$ works since $\int_0^1 k y^{k-1} dy = k \frac{y^k}{k} \Big|_0^1 = 1$

 480E2d20

mine

5) Let Y be a random variable from a distribution with pdf

$$f(y) = 2y \text{ where } 0 < \text{y} < 1.$$

Let $U = 4Y^2$ and find the pdf of U using the method of transformations. Do not forget to include the support of U.

$h(y) = 4y^2, \; h(0) = 0, \; h(1) = 4$

$y^2 = \frac{u}{4}$ so $y = \frac{1}{2} u^{1/2} = h^{-1}(u) = \left(\frac{u}{4}\right)^{\frac{1}{2}}$

$\left|\frac{dh^{-1}(u)}{du}\right| = \frac{1}{4} u^{-\frac{1}{2}} \quad \left(= \frac{1}{2}\left(\frac{u}{4}\right)^{-\frac{1}{2}} \frac{1}{4}\right) = \frac{1}{4\sqrt{u}}$

$f_U(u) = f_Y(h^{-1}(u)) \left|\frac{dh^{-1}(u)}{du}\right| = 2 \frac{1}{2} u^{1/2} \frac{1}{4} u^{-\frac{1}{2}}$

$= \boxed{\frac{1}{4}, \; 0 < u < 4}$

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6) Suppose Y is the number of gypsy moths caught in a trap for the moth. Assume that the mean $\mu = 0.5$ and standard deviation $\sigma = 0.7$. Assume that the sample mean $\overline{Y}$ is computed from a sample of size $n = 50$ traps and that the CLT holds. Find $P(\overline{Y} \geq 0.7)$. 480 Fin22

$\mu_{\overline{Y}} = \mu = 0.5 \qquad \sigma_{\overline{Y}} = \frac{\sigma}{\sqrt{n}} = \frac{0.7}{\sqrt{50}} = 0.09900$

.5 .7

$z = \frac{.7 - .5}{0.09900} = 2.02$

2.02

$\text{prob} = \boxed{0.0217}$

$1 - .0217$ — 8

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$\frac{.7-.5}{.7} = .3059$ — 13

7) Let $Y_n, ..., Y_n$ be a random sample from a uniform distribution on $(\theta, 2\theta)$. Then $E(Y_i) = 3\theta/2$ and $V(Y_i) = \theta^2/12$. Let $T = c\overline{Y}$ be an estimator of θ where c is a constant.

a) Find the bias of T as a function of c and n.

$$ET - \theta = cEY - \theta = \boxed{\frac{3c\theta}{2} - \theta = \theta\left(\frac{3c}{2} - 1\right)}$$

b) Find the mean square error of T as a function of c and n.

$$V(T) = c^2 \frac{V(Y)}{n} = \frac{c^2\theta^2}{12n}$$

$$MSE(T) = V(T) + [B(T)]^2 = \boxed{\frac{c^2\theta^2}{12n} + \left(\frac{3c\theta}{2} - \theta\right)^2}$$

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8) In a 2000 study in the *American Journal of Psychiatry*, investigators wanted to measure the effect of alcohol on the development of the hippocampal region of the brain (the part responsible for long term memory storage). A random sample of 12 adolescents with alcohol abuse disorders was selected. The hippocampal volume in each person was measured, resulting in $\overline{x} = 8.10$ and SD $s = 0.70$. Test whether the mean hippocampal volume in adolescents with alcohol abuse disorders is less than the typical volume of 9.02. (Assume the appropriate test may be used.)

$H_0\ \mu = 9.02 \qquad H_A\ \mu < 9.02$

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{8.1 - 9.02}{0.7/\sqrt{12}} = -4.55$$

$$df = n - 1 = \frac{d}{11} \quad \frac{.005}{-3.106} \quad \text{left tail}$$

$p\text{val} = 0 < .005$

reject H_0 mean hippocampal volume < 9.02 (= typical volume)

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population	n	number admitted with heart disease
1 (men)	1000	52
2 (women)	1000	23

9) Suppose that records of a hospital show that 52 men in a random sample of 1000 male patients versus 23 women in a random sample of 1000 female patients were admitted because of heart disease. Test whether the proportion of men admitted with heart disease is different from the proportion of women admitted with heart disease. Assume that the appropriate 4 step test can be used, and perform the test.

$$\hat{p}_1 = \frac{52}{1000} = 0.052, \quad \hat{p}_2 = \frac{23}{1000} = 0.023$$

$$\hat{p} = \frac{52+23}{1000+1000} = \frac{75}{2000} = 0.0375$$

H_0 $p_1 - p_2 = 0$ $\quad$ H_A $p_1 - p_2 \neq 0$

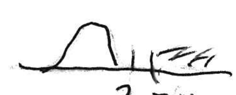
$$z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{.052 - .023}{\sqrt{.0375(1-.0375)\left(\frac{1}{1000}+\frac{1}{1000}\right)}}$$

$$= \frac{.029}{.0084963} = 3.41$$

(pval $= 2(1-.9997) = .0006$)

pval $= 0 < .01$

z	2.576
two tail	.01



reject Ho the prop of men admitted with heart disease is different from the prop of women.

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