Math 483 EXAM 2 covers 2.4, 2.5, 2.7, 2.8, 3.1, 3.2, 3.3, 3.4, 3.8, 3.9, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.9, 5.1, 5.2, and 5.3. The exam is on Thursday, March 6. You are allowed NINE SHEETS OF NOTES and a CALCULATOR. No direct problems on set notation or counting (sections 2.3, 2.6, and 2.9), but you need to know combinations for the binomial distribution. No geometric or Beta distributions (sections 3.5, 4.7). You do not need to know the kernel method technique.

From the 1st exam:

Know what a *sample space* and *event* are.

The *conditional probability* of A *given* B is $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ if $P(B) > 0$.

Know when two events A and B are **independent.**

Complement rule. $P(A) = 1 - P(\overline{A})$.

Know the *additive laws.*

Know P(Y was at least k) = $P(Y \geq k)$ and P(Y at most k) = $P(Y \leq k)$.

The **variance** of Y is $V(Y) = E[(Y - E(Y))^2]$ and the **standard deviation** of Y is SD(Y) = $\sigma = \sqrt{V(Y)}$.

Short cut formula for variance. $V(Y) = E(Y^2) - (E(Y))^2$

If $S_Y = \{y_1, y_2, ..., y_k\}$ then $E(Y) = \sum_{i=1}^{k} y_i p(y_i) = y_1 p(y_1) + y_2 p(y_2) + \cdots + y_k p(y_k)$ and $E[g(y)] = \sum_{i=1}^{k} g(y_i) p(y_i) = g(y_1)p(y_1) + g(y_2)p(y_2) + \cdots + g(y_k)p(y_k)$. Also $V(Y) = \sum_{i=1}^{k} (y_i - E(Y))^2 p(y_i) = (y_1 - E(Y))^2 p(y_1) + (y_2 - E(Y))^2 p(y_2) + \cdots + (y_k - E(Y))^2 p(y_k)$.
Often using $V(Y) = E(Y^2) - (E(Y))^2$ is simpler where $E(Y^2) = y_1^2 p(y_1) + y_2^2 p(y_2) + \cdots + y_k^2 p(y_k)$.

1) **COMMON PROBLEM.** Given a table of y and $p(y)$, find $E[g(Y)]$ and the standard deviation $\sigma = SD(Y)$. See Q2 5, HW5 C, D, E1 2

$E(c) = c$, $E(cg(Y)) = cE(g(Y))$, and $E[\sum_{i=1}^{k} g_i(Y)] = \sum_{i=1}^{k} E[g_i(Y)]$ where c is any constant.

A RV Y is **binomial** if $P(Y = y) = p(y) = \binom{n}{y} p^y q^{n-y}$ for $y = 0, 1, \ldots, n$. Here $q = 1 - p$ and $0 \leq p \leq 1$. $E(Y) = np$, and $V(Y) = np(1 - p)$.

2) **Common Problem.** Given a story problem, recognize that Y is bin(n,p), find E(Y), V(Y), P(Y = y), P(Y is at least j) = $p(j) + \cdots + p(n) = 1 - p(0) - \cdots - p(j-1)$, or P(Y is at most j) = $p(0) + \cdots + p(j) = 1 - p(j+1) - p(j+2) - \cdots - p(n)$. E1 4,6, Q2 2, Q3 3?, HW5 E, F, G, H

MATERIAL NOT ON 1st EXAM

If Y is **Poisson**(λ), then the pmf of Y is $P(Y = y) = p(y) = \dfrac{e^{-\lambda} \lambda^y}{y!}$ for $y = 0, 1, \ldots,$ where $\lambda > 0$. $E(Y) = V(Y) = \lambda$

3) **Common Problem.** Given that Y is a Poisson(λ) RV, find E(Y), V(Y), P(Y = y), P(Y is at least j), P(Y is at most j). Q3 4?, HW6 D, E

The **distribution function** of any RV Y is $F(y) = P(Y \leq y)$ for $-\infty < y < \infty$.

A RV Y is **continuous** if its distribution function $F(y)$ is continuous.

If Y is a continuous RV, then the **probability density function** (pdf) of Y is $f(y) = \frac{d}{dy}F(Y)$ wherever the derivative exists (in this class the derivative will exist everywhere except possibly for a finite number of points). If $f(y)$ is a pdf, then $f(y) \geq 0$ $\forall y$ and $\int_{-\infty}^{\infty} f(t)dt = 1$.

4) **Common Problem.** Find $f(y)$ from $F(y)$. Q4 3, HW6 4.6b

Fact: If Y has pdf $f(y)$, then $F(y) = \int_{-\infty}^{y} f(t)dt$.

5) **Common Problem.** Find $F(y)$ from $f(y)$. HW6 Jb, HW7 Ea.

6) **Common Problem.** Given that $f(y) = c\ g(y)$, find c. Q4 4a?, HW6 Ia, HW7 Ea.

Fact: If Y has pdf $f(y)$, then $P(a < Y < b) = P(a < Y \leq b) = P(a \leq Y < b) = P(a \leq Y \leq b) = \int_a^b f(y)dy = F(b) - F(a)$.

Fact: If Y has a probability function $p(y)$, then Y is discrete and $P(a < Y \leq b) = F(b) - F(a)$, but $P(a \leq Y \leq b) \neq F(b) - F(a)$.

7) **Common Problem.** Given the pdf $f(y)$, find $P(a < Y < b)$, etc. HW6 Hb, Id.

If Y has pdf $f(y)$, then the *mean* of **expected value** of Y is $E(Y) = \int_{-\infty}^{\infty} yf(y)dy$ and $E[g(Y)] = \int_{-\infty}^{\infty} g(y)f(y)dy$. $V(Y) = \int_{-\infty}^{\infty}(y - E[Y])^2 f(y)dy$.

Short cut formula: $V(Y) = E[Y^2] - (E[Y])^2$. (True for discrete and continuous RV's.) The standard deviation of Y is $SD(Y) = \sqrt{V(Y)}$.

8) **COMMON FINAL PROBLEM.** Given the probability function $p(y)$ if Y is discrete or given the pdf $f(y)$ if Y is continuous, find $E[Y]$, $V(Y)$, $SD(Y)$, and $E[g(Y)]$. The functions $g(y) = y$, $g(y) = y^2$, and $g(y) = e^{ty}$ are especially common. E1 2, Q2 5, Q3 2?, Q4 4b?, HW5 C, D, HW7 A, B, HW9 Aa, C, Db

The **moment generating function** (mgf) of a random variable Y is $m(t) = E[e^{tY}]$. If Y is discrete, then $m(t) = \sum_y e^{ty}p(y)$, and if Y is continuous, then $m(t) = \int_{-\infty}^{\infty} e^{ty}f(y)dy$.

The **kth moment** of Y is $E[Y^k]$. Given the mgf $m(t)$ exists for $|t| < b$ for some constant $b > 0$, find the kth derivative $m^{(k)}(t)$. Then $E[Y^k] = m^{(k)}(0)$. In particular, $E[Y] = m'(0)$ and $E[Y^2] = m''(0)$.

9) **Common Problem.** Given the mgf $m(t)$, find $E[Y] = m'(0)$ and $E[Y^2] = m"(0)$. Then find $V(Y) = E[Y^2] - (E[Y])^2$. Q3 1?, Q4 1,2?, HW6 F, G, HW9 Bb, Dc

If Y is **uniform** (θ_1, θ_2) then $f(y) = \dfrac{1}{\theta_2 - \theta_1}$, $\theta_1 \leq y \leq \theta_2$, $E(Y) = (\theta_1 + \theta_2)/2$, and $V(Y) = (\theta_2 - \theta_1)^2/12$.

10) **Common Problem.** Find $f(y)$, $E[g(Y)]$, $P(a < Y < b)$ and $P(a < Y < b|c < Y < d)$ if Y is uniform. HW6 Id, HW7 E, F, G.

Suppose that Y is a RV and that $E(Y) = \mu$ and standard deviation $\sqrt{V(Y)} = SD(Y) = \sigma$ exist. Then the **z-score** is $Z = \dfrac{Y - \mu}{\sigma}$. Note that $E(Z) = 0$, and $V(Z) = 1$.

If Y is **normal** $N(\mu, \sigma^2)$, then the pdf of Y is

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(y-\mu)^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and μ and y are real. $E(Y) = \mu$ and $V(Y) = \sigma^2$.

You need to know how to use the standard normal table for exams 2, 3, 4 and the final.

11) **Common Problem.** Perform a **forwards calculation** using the normal table. In the story problem you will be told that Y is approximately normal with some mean μ and standard deviation σ. You will be given one or two y values and asked **to find a probability** or proportion. Draw a line and mark down the mean and the y values. Find the z-score $z = (y - \mu)/\sigma$. The normal table gives $P(Z > z)$ provided that $z > 0$. Notice that $P(Z < z)$ is given by $1 -$ the table value if $z > 0$. $P(z_1 < Z < z_2)$ is given by the larger table value $-$ the smaller value if $0 < z_1 < z_2$. Use the symmetry of the standard normal curve about 0 and the fact that the total area under the curve is 1 to do more complicated problems: $P(Z > z) = P(Z < -z)$. To use the normal table given $z > 0$, , use the leftmost column and top row of the table to approximate z to two digits. Intersect this row and column to get a 4 digit decimal which is equal to $P(Z > z)$.
See Q4 5a?, Q5?, HW8 B, C, D, Ea

12) **Common Problem.** Perform a **backwards calculation** using the normal table. Here you are **given a probability and asked to find** one or two y_o values. The normal table gives areas to the **right** of z. So if you are asked to find the top 5%, that is the same as finding the bottom 95%. The table can be used to find z_o by finding the 4 digit table number closest to 0.0500. If you are asked to find the bottom 25%, that is the same as finding the top 75%. The table can be used to find $-z_o$ by finding the 4 digit table number closest to 0.2500. If you are asked to find the two values containing the middle 95%, then 5% of the area is outside of the middle. Hence .025 area is to the left of $-z_o$ and $.025 + .95 = .975$ area is to the left of z_o. The table can be used to find z_o by finding the 4 digit table number closest to 0.0250. Suppose the 4 digit number that corresponds to z_o (rather than $-z_o$) has been found. Go along the row to the entry in the leftmost column of the table and go along the column to the top row of the table. For example, if your 4 digit number is .9750, $z_o = 1.96$. To get the corresponding y_o, use $y_o = \mu + \sigma z_o$.
See Q4 5b?, Q5, HW8 A, Eb, and F.

If Y is **exponential** (β) then the pdf of Y is $f(y) = \frac{1}{\beta} \exp(-\frac{y}{\beta}), y \geq 0$ where $\beta > 0$. The distribution function of Y is $F(y) = 1 - \exp(-y/\beta),\ y \geq 0$. $E(Y) = \beta$, and $V(Y) = \beta^2$. If W is exponential (β) then W is gamma $(\alpha = 1, \beta)$.

13) **Common Problem.** Find $f(y)$, $E[g(Y)]$, $P(a < Y < b)$ and $P(a < Y < b | c < Y < d)$ if Y is exponential β. HW8 G

Integration. Know how to do u-substitution (and integration by parts).

If Y is **gamma** (α, β) then the pdf of Y is $f(y) = \dfrac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)},\ y \geq 0$, where α and β, are positive. $E(Y) = \alpha\beta$. $V(Y) = \alpha\beta^2$. If W is χ^2_p, then W is gamma$(\alpha = p/2, \beta = 2)$.

Let Y_1 and Y_2 be discrete random variables. Then the **joint probability function** $p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2)$ and is often given by a table.

The function $p(y_1, y_2)$ is a probability function if $p(y_1, y_2) \geq 0$, $\forall y_1, y_2$ and if $\displaystyle\sum_{(y_1,y_2):p(y_1,y_2)>0} p(y_1, y_2) = 1$.

The **joint distribution function** of any two random variables Y_1 and Y_2 is $F(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2), \forall y_1, y_2$.

Let Y_1 and Y_2 be continuous random variables. Then the **joint probability density function** $f(y_1, y_2)$ satisfies $F(y_1, y_2) = \int_{-\infty}^{y_2} \int_{-\infty}^{y_1} f(t_1, t_2) dt_1 dt_2 \quad \forall y_1, y_2$.

The function $f(y_1, y_2)$ is a joint pdf if $f(y_1, y_2) \geq 0, \forall y_1, y_2$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 dy_2 = 1$.

$P(a_1 < Y_1 < b_1, a_2 < Y_2 < b_2) = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(y_1, y_2) dy_1 dy_2$

$F(y_1, ..., y_n) = P(Y_1 \leq y_1, ..., Y_n \leq y_n)$. In the discrete case, the multivariate probability function is $p(y_1, ..., y_n) = P(Y_1 = y_1, ..., Y_n = y_n)$. In the continuous case, $f(y_1, ..., y_n)$ is a joint pdf if $F(y_1, ..., y_n) = \int_{-\infty}^{y_n} \cdots \int_{-\infty}^{y_1} f(t_1, ..., t_n) dt_1 \cdots dt_n$.

14) **Common Problem.** If $p(y_1, y_2)$ is given by a table, the marginal probability functions are found from the row sums and column sums and the conditional probability functions are found with the above formulas. HW10 Aa, B

15) **COMMON FINAL PROBLEM.** Given the joint pdf $f(y_1, y_2) = kg(y_1, y_2)$ on its support, find k, find the marginal pdf's $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$ and find the conditional pdf's $f_{Y_1|Y_2=y_2}(y_1|y_2)$ and $f_{Y_2|Y_1=y_1}(y_2|y_1)$. See Q5, HW10 C, D, E.

Often using **symmetry** helps.

The *support* of the conditional pdf can depend on the 2nd variable. For example, the support of $f_{Y_1|Y_2=y_2}(y_1|y_2)$ could have the form $0 \leq y_1 \leq y_2$.

Use the following pages for exams 2, 3, 4 and the final.

For **any event** A, $0 \le P(A) \le 1$.

The **probability function** of a discrete RV Y is $p(y) = P(Y = y)$. $0 \le p(y) \le 1$ and $\sum_{y:p(y)>0} p(y) = 1$.

Let Y be a discrete RV with probability function $p(y)$. Then the *mean* or **expected value** of Y is $E(Y) = \mu = \sum\limits_{y:p(y)>0} y\ p(y)$. If $g(Y)$ is a real valued function of Y, then $g(Y)$ is a random variable and $E[g(Y)] = \sum\limits_{y:p(y)>0} g(y)\ p(y)$.

The **variance** of Y is $V(Y) = E[(Y - E(Y))^2]$ and the **standard deviation** of Y is SD(Y) $= \sigma = \sqrt{V(Y)}$. *Short cut formula for variance.* $V(Y) = E(Y^2) - (E(Y))^2$

The **moment generating function** (mgf) of a random variable Y is $m(t) = E[e^{tY}]$. If Y is discrete, then $m(t) = \sum_y e^{ty} p(y)$, and if Y is continuous, then $m(t) = \int_{-\infty}^{\infty} e^{ty} f(y) dy$.

Derivatives. The **product rule** is (f(y)g(y))' = f'(y)g(y) + f(y)g'(y). The **quotient rule** is $(\frac{n(y)}{d(y)})' = \frac{d(y)n'(y) - n(y)d'(y)}{[d(y)]^2}$. Know how to find 2nd, 3rd, etc derivatives. The **chain rule** is [f(g(y))]' = [f'(g(y))][g'(y)]. Know the derivative of ln y and e^y and know the chain rule with these functions. Know the derivative of y^k.

The **distribution function** of any RV Y is $F(y) = P(Y \le y)$ for $-\infty < y < \infty$. If $F(y)$ is a distribution function, then $F(-\infty) = 0$, $F(\infty) = 1$, F is a nondecreasing function, and F is right continuous. $P(a < Y \le b) = F(b) - F(a)$.

Double Integrals. If the region of integration Ω is bounded on top by the function $y_2 = \phi_T(y_1)$, on the bottom by the function $y_2 = \phi_B(y_1)$ and to the left and right by the lines $y_1 = a$ and $y_2 = b$ then $\int \int_\Omega f(y_1, y_2) dy_1 dy_2 = \int \int_\Omega f(y_1, y_2) dy_2 dy_2 =$

$$\int_a^b [\int_{\phi_B(y_1)}^{\phi_T(y_1)} f(y_1, y_2) dy_2] dy_1.$$

Within the inner integral, treat y_2 as the variable, anything else, including y_1, is treated as a constant.

If the region of integration Ω is bounded on the left by the function $y_1 = \psi_L(y_2)$, on the right by the function $y_1 = \psi_R(y_2)$ and to the top and bottom by the lines $y_2 = c$ and $y_2 = d$ then $\int \int_\Omega f(y_1, y_2) dy_1 dy_2 = \int \int_\Omega f(y_1, y_2) dy_2 dy_2 =$

$$\int_c^d [\int_{\psi_L(y_2)}^{\psi_R(y_2)} f(y_1, y_2) dy_1] dy_2.$$

Within the inner integral, treat y_1 as the variable, anything else, including y_2, is treated as a constant.

The **support** of continuous random variables Y_1 and Y_2 is where $f(y_1, y_2) > 0$. The support (plus some points on the boundary of the support) is generally given by one to three inequalities such as $0 \le y_1 \le 1$, $0 \le y_2 \le 1$, and $0 \le y_1 \le y_2 \le 1$. For each variable, set the inequalities to equalities to get boundary lines. For example $0 \le y_1 \le y_2 \le 1$ yields 5 lines: $y_1 = 0, y_1 = 1$, $y_2 = 0, y_2 = 1$, and $y_2 = y_1$. Generally y_2 is on the vertical axis and y_1 is on the horizontal axis for pdf's.

To determine the **limits of integration**, examine the **dummy variable used in the inner integral**, say dy_1. Then within the region of integration, draw a line parallel to the same (y_1) axis as the dummy variable. The limits of integration will be functions of the other variable (y_2), never of the dummy variable (dy_1).

If Y_1 and Y_2 are discrete RV's with joint probability function $p(y_1, y_2)$, then the **marginal probability function for** Y_1 is

$$p_{Y_1}(y_1) = \sum_{y_2} p(y_1, y_2)$$

where y_1 is held fixed. The **marginal probability function for** Y_2 is

$$p_{Y_2}(y_2) = \sum_{y_1} p(y_1, y_2)$$

where y_2 is held fixed. The **conditional probability function of** Y_1 **given** $Y_2 = y_2$ is

$$p_{Y_1|Y_2=y_2}(y_1|y_2) = \frac{p(y_1, y_2)}{p_{Y_2}(y_2)}.$$

The **conditional probability function of** Y_2 **given** $Y_1 = y_1$ is

$$p_{Y_2|Y_1=y_1}(y_2|y_1) = \frac{p(y_1, y_2)}{p_{Y_1}(y_1)}.$$

If Y_1 and Y_2 are continuous RV's with joint pdf $f(y_1, y_2)$, then the **marginal probability density function for** Y_1 is

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 = \int_{\phi_B(y_1)}^{\phi_T(y_1)} f(y_1, y_2) dy_2$$

where y_1 is held fixed (get the region of integration, draw a line parallel to the y_2 axis and use the functions $y_2 = \phi_B(y_1)$ and $y_2 = \phi_T(y_1)$ as the lower and upper limits of integration). The **marginal probability density function for** Y_2 is

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 = \int_{\psi_L(y_2)}^{\psi_R(y_2)} f(y_1, y_2) dy_1$$

where y_2 is held fixed (get the region of integration, draw a line parallel to the y_1 axis and use the functions $y_1 = \psi_L(y_2)$ and $y_1 = \psi_R(y_2)$ as the lower and upper limits of integration). The **conditional probability density function of** Y_1 **given** $Y_2 = y_2$ is

$$f_{Y_1|Y_2=y_2}(y_1|y_2) = \frac{f(y_1, y_2)}{f_{Y_2}(y_2)}$$

provided $f_{Y_2}(y_2) > 0$. The **conditional probability density function of** Y_2 **given** $Y_1 = y_1$ is

$$f_{Y_2|Y_1=y_1}(y_2|y_1) = \frac{f(y_1, y_2)}{f_{Y_1}(y_1)}$$

provided $f_{Y_1}(y_1) > 0$.