1] p23: A **set** is a collection of distinct **elements**:
Notation: use capital letters and braces $A=\{1,2,3\}$.

2] * The **universal set** S is the set of all elements under consideration.

ex) Flip coin $S=\{H,T\}$. Roll die $S=\{1,2,3,4,5,6\}$.

3] * If every element in A is also in B, then A is a **subset of B**; $A \subset B$.
The **empty set** $\emptyset$ is the set with no elements

4] * The **union** of A and B is the set of all points in A or B or both:
$A \cup B = \{x \mid x \text{ is in } A \text{ or } B\}$
inclusive or

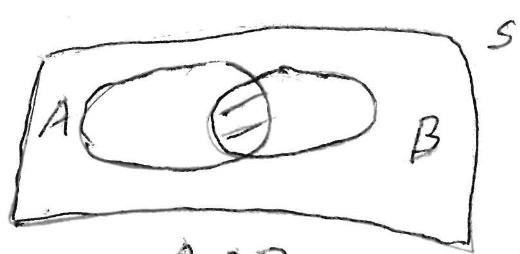
5] p24 * The **intersection** of A and B is the set of points in both A and B:
$A \cap B = \{x \mid x \text{ is in } A \text{ and } x \text{ is in } B\}$

6] * If $A \subset S$, then the **complement** of A $= \bar{A}$, is the set of points in S but not in A
$\bar{A} = \{x \in S \mid x \notin A\}$.

7] p23 **venn diagram** Draw a box for S and circles for sets of interest, often the shaded region denotes the set of interest.

ex]

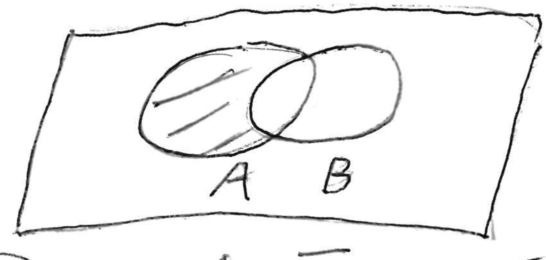
$\overline{A}$ ← shaded



$A \cap B$

← 4 disjoint regions

$A \cup B$



$A \cap \overline{B}$

8] * p24 A and B are disjoint or mutually exclusive

if $A \cap B = \phi$.

Rules: List every element once, order "does not matter".

9] i) $A \cup \overline{A} = S$

distributive laws ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

10] * De Morgan's Laws $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A \cup B} = \underbrace{\overline{A} \cap \overline{B}}_{\text{RHS both } \overline{A} \text{ and } \overline{B} \text{ are used}}$

If LHS has $\cap$, RHS has $\cup$.
If LHS has $\cup$, RHS has $\cap$.

ex] $A = \{1,2\}$, $B = \{2,3\}$, $S = \{1,2,3,4,5\}$.

Then $A \subset S$, $\overline{A} = \{3,4,5\}$, $A \cap B = \{2\}$,

$A \cup B = \{1,2,3\}$.

11] p27 An experiment is the process by which an observation is made

12] p27 A sample point is a possible outcome from an experiment.

13] p28 * The sample space S is the set of all possible outcomes (sample points).

ex] Toss coin once $S = \{H, T\}$

twice $S = \{HH, HT, TH, TT\}$

ex] Machine makes 1000 items per day. Draw every 100th item from the production.

expt 1] check whether item meets specifications or not

$S_1 = \{$10 tuples, each entry a Y or N of the form YYY NNN YNNY$\}$

expt 2] Count how many of the 10 items selected meet specs $S_2 = \{0, 1, \dots, 10\}$.

14]* An event is a subset of S.

15] p27 A simple event cannot be decomposed and consists of exactly one sample point.

16] p28 * A discrete sample space consists of a finite or countable number of outcomes.

17] If S is discrete, an event is any subset of S.

18] p20, 29 Probability is synonymous with chance, odds and likelihood. The relative frequency interpretation of probability says repeat experiment many times. Find the proportion of times each outcome occurs. Then the probability of outcome $E_1 = P(E_1)$ tends towards the proportion of times E_1 would occur if the experiment was done infinitely often.

2.5

ex] toss coin, $S = \{H, T\}$, $P(H) = \frac{1}{2}$

roll die: $P(5) = \frac{1}{6}$

19] Know For any event A, $0 \le P(A) \le 1$.

20] * p 30 The set valued function $P(\ )$ satisfies axioms

1) $P(A) \ge 0$

2) $P(S) = 1$

3) If $A_1, A_2, \ldots,$ are pairwise mutually exclusive events ($A_i \cap A_j = \emptyset$, $i \ne j$),

then $P(A_1 \cup A_2 \cup \ldots) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

Note] If $P(A) = 0$, A is "impossible"

If $P(A) = 1$, A is certain to occur.

Hence $P(S) = 1$.

ex] roll die: $A = \{\text{die was } 7\}$, $S = \{1, 2, 3, 4, 5, 6\}$

so $P(A) = 0$.

p30

21] * Addition law for mutually exclusive events

If $A_1, A_2, \ldots, A_n$ are pairwise mutually exclusive, then $P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i)$.

(In particular $P(A) = 1 - P(\bar{A})$.)

22] A and $\bar{A}$ are mutually exclusive.

If $S = \{E_1, \ldots, E_K\}$, then $E_1, \ldots, E_K$ are pairwise mut. excl.

23] For now, "randomly selected" items are equally likely to be selected. [illegible] select 1, $P(\text{ith student is selected}) = \frac{1}{35}$ [illegible]

24] To list a sample space, use order. (common problem) 483 3

Flip coin twice

TT
TH
HT
HH

three times p37

TTT	HTT
TTH	HTH
THT	HHT
THH	HHH

Toss die twice

1st \ 2nd	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2						
3						
4						
5				(5,4)		
6						

(5,4): 1st=5, 2nd=4

etc
36 outcomes
equally likely for fair die

Finding the prob of an event via the Sample point method

25]* p30, 34 Suppose that $S = \{E_1, E_2, \dots, E_k\}$.

Then $0 \le P(E_i) \le 1$ and $\sum_{i=1}^{k} P(E_i) = 1$.

event $E_i = \{E_i\}$

An event A is a subset of S, so

$$P(A) = \sum_{i : E_i \in A} P(E_i).$$

That is, $P(A)$ is the sum of the probabilities of the sample points in A.

ex] Flip coin 3 times A = 2 or more Heads

$$P(A) = P\{THH, HTH, HHT, HHH\} = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$$

ex] Common problem leave the probabilities for some outcomes blank

grade	A	B	C	D	F
Prob	.2	.3	.2		

Find Prob of D or F

$1 - .2 - .3 - .2 = .3$

Read examples 2.2, 2.3, 2.4 and the 1st paragraph on 37 carefully.

3.5

26] * p37 Suppose that all of the sample points E_i in S are equally likely and that $S = \{E_1, E_2, \ldots, E_K\}$. Then $P(E_i) = \frac{1}{K}$ and if A is an event, then $P(A) = \frac{\#\ E_i \text{ in } A}{\#\ E_i \text{ in } S} = \frac{\#\ E_i \text{ in } A}{K}$,

where $\#\ E_i$ in A = # of sample points in A.

Note] This situation occurs with fair coins and fair die, but not in the last ex.

Section §2.6 p41

27] *The multiplication (mn) rule. An experiment is performed in k parts $G_1, G_2, \ldots, G_K$. The number of possible outcomes for part $G_i = n_i$. Then the total experiment consists of performing parts G_1 then G_2 then ... then G_K and has

$\underbrace{n_1 \cdot n_2 \cdot \cdots n_K}_{\text{multiply}}$ possible outcomes.

Technique 1: use slots to list the parts

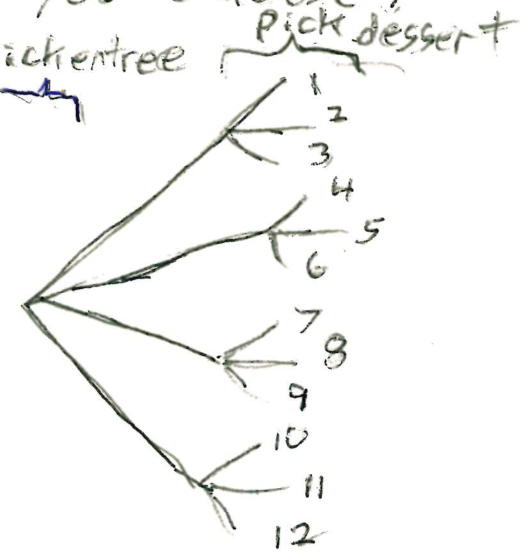
$$\frac{n_1}{G_1} \cdot \frac{n_2}{G_2} \cdots \frac{n_k}{G_K}$$

Technique 2: For small experiments use tree diagrams.

ex] A cafeteria offers 4 entrees and 3 desserts. You may choose one entree and one dessert. How many different meals can you choose?

soln) $\underset{E}{\underline{4}} \cdot \underset{D}{\underline{3}} = 12$ or

tree diagram →



ex] A combination lock has 4 tumblers, each with the digits 0, ..., 9. How many combinations are possible?

$$\underset{1}{\underline{10}} \cdot \underset{2}{\underline{10}} \cdot \underset{3}{\underline{10}} \cdot \underset{4}{\underline{10}} = \underbrace{10^4}_{\text{exponential notation}} = 10000$$

Read p41-2 carefully, but ex 2.7 is too hard.

28] p43 * An ordered arrangement of r distinct objects is a <u>permutation</u>. The number of ways of ordering n distinct objects taken r at a time is $P_r^n = n(n-1)\cdots(n-r+1)$

$$= \frac{n!}{(n-r)!}.$$

Here $n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1.$

$n! = n\cdot(n-1)! = n\cdot(n-1)\cdot(n-2)! = n\cdot(n-1)\cdot(n-2)\cdot(n-3)!$ etc

$3! = 3\cdot 2\cdot 1 = 6$, $1! = 1$, $0! = 1$ by convention

4.5

<u>Proof</u> that $P^n_r = n \cdot (n-1) \cdots (n-r+1)$.

A permutation fills r positions with n distinct objects. use the multiplication (mn) rule

$$\underset{1st}{\underline{n}} \quad \underset{2nd}{\underline{n-1}} \qquad \underbrace{\underset{rth}{\underline{n-r+1}}}_{}$$

r-1 objects havebeen used; $n-(r-1)$ remain

---

ex] $P^n_n = n!$

ex] Consider an expt in which a person is selected from a population of size n, a 2nd person is selected from the remaining n-1, and a 3rd person from the remaining n-2. This process is called <u>sampling without replacement</u>

Each possible sample of size r is a permutation.

ex] Select 3 students from a class of size 30 without replacement. How many samples are there?

Soln] $P^{30}_3 = 30 \cdot 29 \cdot 28 = 24360$

ex] Now suppose that an object is selected from a pop. of size n, then this object is placed back into the pop and another object selected (it is possible that the same object is selected again). This process is called <u>sampling with replacement</u>. The number of possible samples of size r $= \underset{1}{n} \; \underset{2}{n} \cdots \underset{r}{n} = n^r$

ex] Select 3 students from a class of size 30 with replacement. There are $(30)^3 = 27000$ possible samples.

ex] choosing officers: club has 25 members. want a president and a secretary. How many ways can these positions be filled?

$$P_2^{25} = \underset{P}{\underline{25}}\ \underset{S}{\underline{24}} = 600$$

ex) Arranging books: Six different books are to be arranged on a shelf. How many arrangements are there?

$$\underset{1}{\underline{6}}\ \underset{2}{\underline{5}}\ \underset{3}{\underline{4}}\ \underset{4}{\underline{3}}\ \underset{5}{\underline{2}}\ \underset{6}{\underline{1}} = 6! = P_6^6 = 720$$

29] p44 Let N be the number of ways of partitioning n distinct objects into k distinct groups containing $n_1, n_2, \dots, n_k$ objects respectively where each object is in exactly one group and $\sum_{i=1}^{k} n_i = n$. Order is ignored in each group.

$$\text{Then } N = \underbrace{\binom{n}{n_1\ n_2 \cdots n_k}}_{\text{multinomial coefficient}} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

This is sometimes called a permutation with repetition

30] Multinomial theorem

$$(y_1 + y_2 + \dots + y_k)^n = \sum \binom{n}{n_1\, n_2 \cdots n_k} y_1^{n_1} y_2^{n_2} \cdots y_k^{n_k}$$

where the sum is taken over all $n_i = 0, 1, \dots, n$ such that $\sum_{i=1}^{k} n_i = n$.

ex] How many different 11-letter words (real or imaginary) can be formed from the word mississippi? (5.5)

Soln) Want the number of distinct 11-letter words with 4 i's, 4 s's, 2 p's and 1 m

so $\dfrac{11!}{4!\,4!\,2!\,1!} = 34650.$

11 objects (letters)
4 groups
i's, s's, p's, m's

ex] How many ways can 5 green, 4 blue and 3 red bulbs be arranged in a string of (eg Christmas tree) lights with 12 sockets?

Soln: objects are sockets, 3 groups

so $\dfrac{12!}{5!\,4!\,3!}$

31] p46 * The number of unordered subsets of size r (chosen without replacement) from n objects is $\binom{n}{r} = \dfrac{n!}{r!(n-r)!} = C_r^n = \dfrac{P_r^n}{r!}$.

(bin coef = n choose r)

32] When the order of the selected objects is unimportant, the r selected (distinct) objects are called a <u>combination</u> and there are $\binom{n}{r}$ combinations. When the order of the r objects is important, the r selected objects are called a permutation & there are P_r^n permutations.

33] p46 * Binomial theorem $(x+y)^n = \sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\text{binomial coefficient}} x^{n-i} y^i$

Pick 1 person to come up and wave

ex] $\binom{n}{0} = \dfrac{n!}{0!\,n!} = 1$, $\binom{n}{1} = \dfrac{n!}{1!(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$

$\binom{n}{r} = \binom{n}{n-r}$, $\binom{n}{n-1} = n$, $\binom{n}{n} = 1.$

ex] A, B, C

	president P	secretary S
1	A	B
2	B	A
3	A	C
4	C	A
5	B	C
6	C	B

← order matters

Combine each of the $r!$ permutations of the same r members to get a combination of the r members in $C_r^n = P_r^n / r!$ ways

← Permute each combination of r members in $r! = 2! = 2$ ways to get the $P_r^n = r!\,C_r^n$ permutations

1, 2 ⟷ AB

3, 4 ⟷ AC

5, 6 ⟷ BC

$3 \cdot 2 = 6$ ways

$= P_2^3 = \frac{3!}{(3-2)!}$

permutations, order matters

$C_2^3 = \frac{3!}{2!\,1!} = 3$ ways

combination order does not matter

ex] Select a committee of 5 members from 100 senators in $C^{100}_{5} = \binom{100}{5}$

$$= \frac{100!}{5!\,95!} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 \cdot (95!)}{5!\ 95!} = 75287520$$

ex] A fair coin is tossed 10 times. What is the probability of obtaining 3 or fewer heads?

better way later binomial RV

soln] Let $S = \{$ sequences of 10 coin flips $\}$.

S contains $\underset{1st}{\underline{2}} \cdots \underset{10th}{\underline{2}} = 2^{10}$ elements.

Let A be the event of 3 or fewer heads, and let $E_i = \{ i$ heads in 10 tosses $\}$, $i = 0, 1, 2, \ldots, 10$.

Then $P(A) = P(E_0) + P(E_1) + P(E_2) + P(E_3)$

$$= \frac{\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3}}{2^{10}} = \frac{1 + 10 + 45 + 120}{1024}$$

$= \frac{176}{1024} = .1719$. Note that $\binom{10}{k}$ is the number of sequences that contain k heads.

ex] A standard deck of cards consists of 52 cards in 4 suits, Hearts, Diamonds, Clubs and Spades of 13 cards each:

$2, 3, \ldots, 10, \overbrace{\text{Jack, Queen, King}}^{\text{face cards}}$, Ace.

Hearts and diamonds are red, clubs and spades black.

Suppose a person is dealt a five card hand.

what is the probability that all 5 cards are spades? 6.5

soln]
$$\frac{\binom{13}{5}}{\binom{52}{5}} = \frac{1287}{2598960} = \frac{C^{13}_5}{C^{52}_5} =$$

$$\frac{13\cdot 12\cdot 11\cdot 10\cdot 9}{52\cdot 51\cdot 50\cdot 49\cdot 48} = \frac{P^{13}_5}{P^{52}_5} = .0004952.$$

*Note: If the number of elements in S is found without using order: (denominator), then the number of desired hands should also be found without using order (numerator).

ex) Let A_2 be the set of 5 card hands that contain exactly 2 kings, 2 queens and one jack.

$$P(A_2) = \frac{\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}}.$$

mult rule $\overline{2K}\ \overline{2Q}\ \overline{1J}$

ex) A lot consists of 100 fuses, 5 are chosen at random. If all 5 work, the lot is accepted. Assume 20 fuses will not work. What is the prob that the lot is accepted?

Let A denote the event that the lot is accepted.

$$P(A) = \frac{\binom{20}{0}\binom{80}{5}}{\binom{100}{5}} \approx .32.$$

$\leftarrow$ # of samples of size 5 with 0 defectives

$\underline{\text{no defectives}}$ $\underline{\text{all work}}$

mult rule

$\leftarrow$ total number of samples of size 5

ex] Find # of 5 card hands with exactly one pair ?
(no 2 pair, no 3 of a kind, etc).

soln $\underline{13} \cdot \underline{\binom{4}{2}} \cdot \underline{\binom{12}{3}} \cdot \underline{4} \cdot \underline{4} \cdot \underline{4}$

$13 = \binom{13}{1}$; pick pair denomination (eg ace)

$4 = \binom{4}{1}$

get 2 cards from that denomination

get the other 3 denominations (eg Jack, Queen, King)

card from 1st denom

" " " " 2nd

" " " " 3rd

$4 = \binom{4}{1}$

$= 1{,}098{,}240$

so $P(\text{pair}) = \dfrac{1098240}{\binom{52}{5}}$

The important point is that the number of card denominations has been reduced from 13 to 12.

ex] Deal 2 cards. Find the # of ways that the 1st is a King and the 2nd is a King

$\underset{1st}{\underline{\binom{4}{1}}}\ \underset{2nd}{\underline{\binom{3}{1}}} = 12 = P^4_2 = 4 \cdot 3$ ← 3 Kings left

§2.7 34] Know p 52 The conditional probability of event A given that event B has occurred is $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ if $P(B) > 0$.

Read the relative freq. interpretation on P. 52.

35] Think of a conditional probability as an experiment with sample space B instead of S.

(The expt where only outcomes in B are of interest.)

ex) F 2001 math 150

Section	grade A	B	C	D	F	row total
M	11	8	3	1	2	25
all others	19	41	40	1	37	138
column total	30	49	43	2	39	163

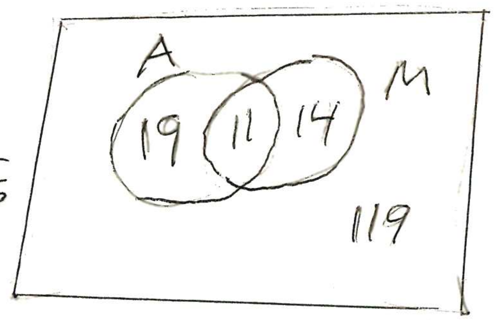
← grand total

a) Find the probability that a randomly selected student got an A.

$$P(A) = \frac{30}{163} = .184$$

b) Find the prob that a randomly selected student got an A given that the student was in section M.

$$P(A|M) = \frac{P(A \cap M)}{P(M)} = \frac{\frac{11}{163}}{\frac{25}{163}} = \frac{11}{25}$$



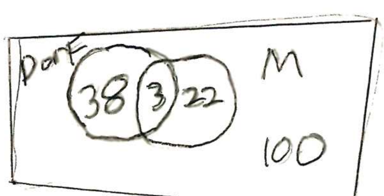
$$= \frac{11}{11+14} = .44$$

$\frac{11}{25}$ ← row M acts as new sample space

c) Find the prob that a randomly selected student got a D or F.

$$P(D \cup F) = \frac{2+39}{163} = .251$$

d) Find the prob that a randomly selected student got a D or F given that the student was in section M.



$$P(D\text{or}F|M) = \frac{P(D\text{or}F \cap M)}{P(M)} = \frac{\frac{3}{163}}{\frac{25}{163}}$$

$$= \frac{3}{25} = \frac{3}{3+22} = 0.12$$

36] Common Problem

You are given a table with i rows and j columns and asked to find unconditional probabilities and conditional probabilities. Find the row, column, and grand totals and proceed as in the previous example.

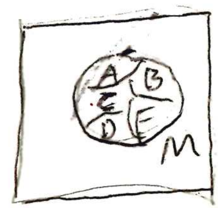
Note: $P(A)$ was an unconditional problem while $P(A|M)$ was conditional.

ex] $P(M|A) = \frac{11}{30} \approx .367$ ← column A acts as the new sample space

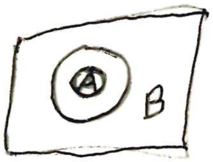
$P(M) = \frac{25}{163} = .153$

$$\underbrace{\begin{array}{ccccc} P(A|M) & P(B|M) & P(C|M) & P(D|M) & P(F|M) \\ \frac{11}{25} & \frac{8}{25} & \frac{3}{25} & \frac{1}{25} & \frac{2}{25} \end{array}}$$

these sum to one, a conditional prob $P(\cdot|M)$ is a prob with sample space M.



37] Note: if $A \subseteq B$, then $P(A \cap B) = P(A)$



$A \subseteq B$

see ex 2.14.

38] p53] * Two events A and B are <u>independent</u> if <u>any</u> <u>one</u> of the following 3 conditions hold

I1) $P(A \cap B) = P(A)\,P(B)$

I2) $P(A|B) = P(A)$

or I3) $P(B|A) = P(B)$.

Otherwise, A and B are <u>dependent</u>.

39] Interpretation: If $P(A)$ is unaffected by the occurrence or nonoccurrence of event B, 8.5

then A and B are independent.

Fact] If A and B are ind, then so are $\bar{A}$ and B, $\bar{A}$ and $\bar{B}$, and A and $\bar{B}$.

ex] A = Your HW grade B = it rained today in England.

A and B are independent

40] $\boxed{\text{common problem}}$ Given some of the following probabilities $P(A)$, $P(B)$, $P(A \cap B)$ and $P(A|B)$, find $P(A|B)$ and whether A and B are independent.

ex] $P(A) = .4$, $P(B) = .3$, $P(A \cap B) = .12$,

then $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{.12}{.3} = .4 = P(A)$

$P(A \cap B) = P(A)P(B)$ so A and B are independent.

41] * Suppose that $P(A) > 0$ and $P(B) > 0$. If A and B are disjoint, then knowing that A occurred means that B did not occur. Being mutually exclusive is an extreme form of dependence. $P(A \cap B) = 0$ $P(A|B) = 0$

42] $\boxed{\text{common problem}}$: given a table, determine if a row event and column event are independent.

ex]

	A	B	C	D	F	row tot
M	11	8	3	1	2	25
other	19	41	40	1	37	138
col tot	30	49	43	2	39	163

Are M and A ind? $P(M) = \frac{25}{163}$, $P(A) = \frac{30}{163}$, $P(A \cap M) = \frac{11}{163}$

$\neq P(M)P(A)$ not ind