Note] *

if $n \le 30$
t interval
$\bar{Y} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$
$df = n-1 \le 29$

if $n > 30$.
t interval
$\underbrace{\bar{Y} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}}_{\text{use if } n > 30}$

If $n > 30$, then $df = n-1 > 29$ and t table df only goes to 29.

The line $df = \infty$ is used if $df > 29$ and this line gives $z_{\alpha/2}$.

42] common E4 problem ex]
$\bar{X} = 100$, $S = 40$

a) If $n = 16$ and the data are iid from a highly skewed dist, find a 95% CI for μ if possible.

Soln) not possible

b) If $n = 16$ and CLT holds, find a 95% CI for μ if possible.

Soln] $df = n-1 = 16-1 = 15$

.95 .025

$t_{.025}$	df
2.131	15

see HW 8 F.

95% CI is $\bar{X} \pm t_{\alpha/2} \frac{s}{\sqrt{n}} = 100 \pm 2.131 \frac{40}{\sqrt{16}}$

$= 100 \pm 21.31 = [78.69, 121.31]$

c) If $n=32$ and CLT holds, find 95% CI for μ.

soln] $df = n-1 = 31 > 29$
so use $z_{\alpha/2}$ (go to df = inf

$t_{.025}$	df
1.96	inf.

$$\bar{X} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} = 100 \pm 1.96 \frac{40}{\sqrt{32}}$$

↑ HW18 $= 100 \pm 13.859 = [86.141, 113.859]$

43] Know how to compute

$$\bar{Y} = \frac{1}{n} \Sigma Y_i \quad \text{and} \quad s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2}$$

ex] 1996 wanted to know mean Vitamin C content μ of a corn soy blend.
Find 95% CI for μ if data is

measured in mg per 100g

26 31 23 22 11 22 14 31

see HW19 A

Y_i	$Y_i - \bar{Y}$	$(Y_i - \bar{Y})^2$
26	3.5	12.25
31	8.5	72.25
23	0.5	.25
22	-0.5	.25
11	-11.5	132.25
22	-0.5	.25
14	-8.5	72.25
31	8.5	72.25
$\Sigma Y_i = 180$	$\Sigma Y_i - \bar{Y} = 0$ always, good check	$\Sigma (Y_i - \bar{Y})^2 = 362$

$n = 8$

$\bar{Y} = \frac{180}{8} = 22.5$

So $s^2 = \dfrac{\sum (y_i - \bar{y})^2}{n-1} = \dfrac{362}{7} = 51.714.$

and $s = \sqrt{s^2} = \sqrt{51.714} = 7.191.$

Hence the 95% CI for μ is

$\bar{Y} \pm t_{\alpha/2} \dfrac{s}{\sqrt{n}} = 22.5 \pm 2.365 \dfrac{7.191}{\sqrt{8}}$

$df = n-1 = 7$

1.99 .025

$t_{.025}$	df
2.365	7

$= 22.5 \pm 6.0128$

$= [16.487, 28.513]$

44] p426 Justification for t interval:

If $Y_1, \ldots, Y_n$ are iid $N(\mu, \sigma^2)$,

then $\dfrac{\bar{Y} - \mu}{s/\sqrt{n}} \sim t_{n-1}$. So $1 - \alpha =$

$$P\left(-t_{\alpha/2} < \frac{\bar{Y} - \mu}{s/\sqrt{n}} < t_{\alpha/2}\right) =$$

$$P\left(-t_{\alpha/2} \frac{s}{\sqrt{n}} < \bar{Y} - \mu < t_{\alpha/2} \frac{s}{\sqrt{n}}\right)$$

$$= P\left(-\bar{Y} - t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} < -\mu < -\bar{Y} + t_{\alpha/2} \frac{s}{\sqrt{n}}\right)$$

$$= P\left(\bar{Y} - t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} < \mu < \bar{Y} + t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}\right)$$

45] Common Final problem: find a small sample CI for μ and either a large sample CI for $\mu_1 - \mu_2$ or for $p_1 - p_2$ or for p

46] p428 **The pooled t**

$100(1-\alpha)\%$ CI for $\mu_1 - \mu_2$

is $(\bar{Y}_1 - \bar{Y}_2) \pm t_{\frac{\alpha}{2}} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

where $P(T > t_{\alpha/2}) = \alpha/2$

and $T \sim t$ with $df = n_1 + n_2 - 2$

and $S_p = \sqrt{S_p^2}$ where

pooled variance $S_p^2 = \dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 2}$

If $df = n_1 + n_2 - 2 \geq 30$, use $z_{\alpha/2}$ instead of $t_{\alpha/2}$.

47] p430 Assumptions: a)

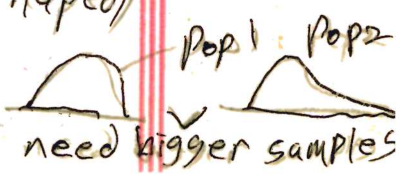
Sample 1 $Y_{11}, \ldots, Y_{1n_1}$ is a random sample from a pop with mean μ_1 and SD σ_1.

Sample 2 $Y_{21}, \ldots, Y_{2n_2}$ is a random sample from a pop with mean μ_2 and SD σ_2.

b) σ_1 and σ_2 are unknown but $\sigma_1 = \sigma_2$ (or $\sigma_1 \approx \sigma_2$). ($n_1 = n_2$ also works)

c) If n_1 and n_2 are both small, this CI works best if $\sigma_1 = \sigma_2$, $n_1 = n_2$ and the 2 underlying populations have the same shape (especially if both are mound shaped)

ex} 

ex} 

ex]

Soluble

Chemical oxygen demand (SCOD) from
14 sludge samples and from another
16 sludge samples using a different type of culture
Get 95% CI for $\mu_1 - \mu_2$ assuming $\sigma_1 = \sigma_2$.

	n	mean	SD	var
pop1	14	18.1	6	36
pop2	16	15.9	5	25

usually only one given

$\mu_1 - \mu_2$ = difference in mean SCOD's

$$S_p^2 = \frac{(n_1-1)\,S_1^2 + (n_2-1)\,S_2^2}{n_1+n_2-2}$$

← using SDs

$$S_p^2 = \frac{(14-1)\,6^2 + (16-1)(5)^2}{14+16-2} = \frac{843}{28} = 30.107$$

$S_p = 5.487$

95% CI $df = n_1+n_2-2 = 28$

d	
28	2.048
	95%

$$\bar{Y}_1 - \bar{Y}_2 \pm t_{\alpha/2}\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} =$$

$$18.1 - 15.9 \pm 2.048 \;\; 5.487 \underbrace{\sqrt{\frac{1}{16} + \frac{1}{14}}}_{0.366} =$$

$$2.2 \pm 4.112 = [-1.912,\ 6.312].$$

Note: 0 is a reasonable value.

48] p435 Let $Y_1, \ldots, Y_n$ be iid $N(\mu, \sigma^2)$. 62.5

Then a $100(1-\alpha)\%$ CI for σ^2 is

$$\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}\right]$$ where $df = n-1$.

and the $100(1-\alpha)\%$ CI for σ is

$$\left[\sqrt{\frac{(n-1)S^2}{\chi^2_{\alpha/2}}}, \sqrt{\frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}}\right].$$

49] Bad CI if data are not iid $N(\mu, \sigma^2)$.

ex] breakdown voltage of electrically stressed circuits $n = 17$, $S^2 = 137324.3$

95% CI for σ^2

table 6 p 850 – 851

df	$\chi^2_{.975}$	$\chi^2_{.025}$
16	6.908	28.845

CI for σ^2 is $$\left[\frac{16(137324.3)}{\underset{\uparrow \text{bigger value}}{28.845}}, \frac{16(137324.3)}{\underset{\uparrow \text{smaller value}}{6.908}}\right]$$

$= [76172.258, 318064.380]$

CI for σ takes $\sqrt{\ }$ of the endpoints, giving $[275.993, 563.972]$

50] common problem

get pooled CI for $\mu_1 - \mu_2$
and CI for σ^2

HW19

§9.6 1] p472 The pop. kth moment is $E(Y^k)$ and the sample kth moment is $\frac{1}{n}\sum_{i=1}^{n} Y_i^k$.

2] p473 Let $\mu_k'(\underset{\sim}{\theta}) = E(Y^k)$

$$m_k' = \frac{1}{n}\sum_{i=1}^{n} Y_i^k.$$

Let $\underset{\sim}{\theta} = [\theta_1, \ldots, \theta_t]$. The method of moments estimators $(\hat{\theta}_1, \ldots, \hat{\theta}_t)$ are found by solving

$$\mu_1'(\underset{\sim}{\theta}) = m_1'$$
$$\mu_2'(\underset{\sim}{\theta}) = m_2'$$
$$\vdots$$
$$\mu_t'(\underset{\sim}{\theta}) = m_t'$$

for $\theta_1, \ldots, \theta_t$ where t is the number of unknown parameters.

not in text: If $g(\underset{\sim}{\theta}) = g(\mu_1'(\underset{\sim}{\theta}), \ldots, \mu_r'(\underset{\sim}{\theta}))$, then the method of moments estimator of $g(\underset{\sim}{\theta})$ is $T = g(m_1', \ldots, m_r')$

ex 9.13] p 474 $Y_1, \ldots, Y_n$ iid Gamma α, β

$E\,Y_i = \alpha\beta$, $V(Y_i) = \alpha\beta^2$

So $\mu_1' = E\,Y_1 = \alpha\beta$, $\mu_2' = E(Y_i^2) = V(Y_i) + (E\,Y_i)^2$
$= \alpha\beta^2 + \alpha^2\beta^2$

63.5

$$\mu_1' = \alpha\beta \overset{set}{=} \bar{Y} \qquad (1)$$

$$\mu_2' = \alpha\beta^2 + \alpha^2\beta^2 \overset{set}{=} \frac{1}{n}\Sigma Y_i^2 \qquad (2)$$

$$\hat{\beta} = \bar{Y}/\hat{\alpha} \qquad \text{from} \quad (1)$$

so $\hat{\alpha}\dfrac{(\bar{Y})^2}{\hat{\alpha}^2} + \hat{\alpha}^2\dfrac{(\bar{Y})^2}{\hat{\alpha}^2} = \dfrac{1}{n}\Sigma Y_i^2$ from(2)

or $\dfrac{(\bar{Y})^2}{\hat{\alpha}} = \dfrac{1}{n}\Sigma Y_i^2 - (\bar{Y})^2$

or $\dfrac{1}{\hat{\alpha}} = \dfrac{\frac{1}{n}\Sigma Y_i^2 - (\bar{Y})^2}{(\bar{Y})^2}$

$$\hat{\alpha} = \frac{(\bar{Y})^2}{\frac{1}{n}\Sigma Y_i^2 - (\bar{Y})^2}$$

so $\hat{\beta} = \bar{Y}\dfrac{1}{\hat{\alpha}} = \dfrac{\frac{1}{n}\Sigma Y_i^2 - (\bar{Y})^2}{\bar{Y}}$

Note $\Sigma(Y_i - \bar{Y})^2 = \Sigma[Y_i^2 - 2Y_i\bar{Y} + (\bar{Y})^2]$

skip

$$= \Sigma Y_i^2 - 2\bar{Y}\Sigma Y_i + n(\bar{Y})^2$$
$$= \Sigma Y_i^2 - 2n(\bar{Y})^2 + n\bar{Y}^2 = \Sigma Y_i^2 - n(\bar{Y})^2$$

so $\frac{1}{n}\Sigma Y_i^2 - (\bar{Y})^2 = \frac{1}{n}[\Sigma Y_i^2 - n(\bar{Y})^2]$

and $\hat{\beta} = \dfrac{\frac{1}{n}[\Sigma Y_i - n(\bar{Y})^2]}{\bar{Y}} = \dfrac{\Sigma Y_i - n(\bar{Y})^2}{n\bar{Y}}$

ex] $g(\underset{\sim}{\theta}) = Var(Y) = EY^2 - (EY)^2 = \mu_2' - (\mu_1')^2$

so method of moments estimator

is $m_2' - (m_1')^2 = \frac{1}{n}\sum Y_i^2 - \left(\frac{1}{n}\sum Y_i\right)^2$.

Note that $\sum (Y_i - \bar{Y})^2 = \sum Y_i^2 - n(\bar{Y})^2$.

So $m_2' - (m_1')^2 = \frac{1}{n} \sum Y_i^2 - (\bar{Y})^2$

$= \frac{1}{n}\left[\sum Y_i^2 - n(\bar{Y})^2\right] = \frac{1}{n}\sum (Y_i - \bar{Y})^2$

$= \frac{n-1}{n} \frac{1}{n-1} \sum (Y_i - \bar{Y})^2 = \frac{n-1}{n} S^2$.

skip (A short cut formula for S^2 is

$$S^2 = \frac{n}{\underbrace{n}_{1}} \frac{\left[\sum Y_i^2 - n(\bar{Y})^2\right]}{n-1} = \frac{n}{n-1}\left[\frac{1}{n}\sum Y_i^2 - (\bar{Y})^2\right]$$

$$= \frac{n}{n-1}\left(m_2' - (m_1')^2\right) \approx EY^2 - (EY)^2$$

for large n.)

3) $\boxed{\text{Common Final Problem}}$ Given $Y_1, \ldots, Y_n$ are iid with mean $\mu = EY = \mu_1'$ and Variance $\sigma^2 = EY^2 - (EY)^2 = \mu_2' - (\mu_1')^2$, find the method of moments estimator of a simple function of μ_1' and μ_2'.

In particular, find $EY = h^{-1}(\theta)$

so $h^{-1}(\theta) \overset{set}{=} \bar{Y}$ and $\hat{\theta} = h(\bar{Y})$.

ex] $Y_1, \dots, Y_n$ are iid normal with mean $= \frac{1}{\theta}$ and variance $= \frac{1}{\theta}$.

Find the method of moments estimator of θ.

Soln] $EY = \frac{1}{\theta} \overset{set}{=} \bar{Y}$

so $\tilde{\theta} = \frac{1}{\bar{Y}}$.

Note that since $V(Y) = \frac{1}{\theta}$,

$\theta > 0$. But it is possible that $\bar{Y} < 0$

so could get $\hat{\theta} < 0$.

↑HW20

p461, 477

§9.7 4] * Let $Y_1, \dots, Y_n$ be iid from a pop with pdf $f(y|\underset{\sim}{\theta})$.

Then the joint pdf of $Y_1, \dots, Y_n$ is $\underbrace{f(y_1, \dots, y_n|\underset{\sim}{\theta})}_{\text{function of } y_1, \dots, y_n} = \prod_{i=1}^{n} f(y_i|\underset{\sim}{\theta})$.

The likelihood function is a function of $\underset{\sim}{\theta}$

and $L(\underset{\sim}{\theta}) = L(\underset{\sim}{\theta}|y_1, \dots, y_n) = \prod_{i=1}^{n} f(y_i|\underset{\sim}{\theta})$.

p461, 477

5] * Let $Y_1, \dots, Y_n$ be iid from a pop with probability function $p(y|\underset{\sim}{\theta})$.

Then the joint prob function is $p(y_1, \dots, y_n|\underset{\sim}{\theta}) = \prod_{i=1}^{n} p(y_i|\theta)$

and the likelihood function $L(\underset{\sim}{\theta}) = \prod_{i=1}^{n} p(y_i|\underset{\sim}{\theta})$.

6] Let $\underset{\sim}{y} = (y_1, \ldots, y_n)$ and $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_K)$.

Let $\Theta = \{\underset{\sim}{\theta} \mid f(\underset{\sim}{y}/\underset{\sim}{\theta}) \text{ is a pdf}\}$

or $\Theta = \{\underset{\sim}{\theta} \mid p(\underset{\sim}{y}/\underset{\sim}{\theta}) \text{ is a probability function}\}$

Then Θ is the <u>parameter space</u>.

ex] $Y_i \sim N(\mu, \sigma^2)$, $\underset{\sim}{\theta} = (\mu, \sigma^2)$

$$\Theta = \{(\mu, \sigma^2) \mid \mu \in \mathbb{R},\ \sigma^2 > 0\}$$

ex] $Y_i \sim \text{poisson}(\lambda)$, $\underset{\sim}{\theta} = \theta = \lambda$

$$\Theta = \{\lambda \mid \lambda > 0\}.$$

7] p477 <u>Know</u> For each sample point $\underset{\sim}{y}$ let $\underset{\sim}{\hat{\theta}}(\underset{\sim}{y}) \in \Theta$ be a parameter value at which $L(\underset{\sim}{\theta}) = L(\underset{\sim}{\theta}/\underset{\sim}{y})$ attains its maximum (as a function of $\underset{\sim}{\theta}$ where $\underset{\sim}{y}$ is treated as a constant). A <u>maximum likelihood estimator</u> (MLE) of $\underset{\sim}{\theta}$ based on a sample $\underset{\sim}{Y}$ is $\underset{\sim}{\hat{\theta}}(\underset{\sim}{Y})$.

8) $\hat{\underset{\sim}{\Theta}}(\underset{\sim}{y}) \in \Theta$, so if you find MLE $\hat{\underset{\sim}{\Theta}} \notin \Theta$, you made a mistake.

9) Notation: Let $\underset{\sim}{\theta} = (\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2)$ and $\hat{\underset{\sim}{\theta}}(\underset{\sim}{Y}) = (\hat{\underset{\sim}{\theta}}_1(\underset{\sim}{Y}), \hat{\underset{\sim}{\theta}}_2(\underset{\sim}{Y}))$. We will say that $\hat{\underset{\sim}{\theta}}_i(\underset{\sim}{Y})$ is the MLE of $\underset{\sim}{\theta}_i$, $i=1,2$.

10] p480 know (Invariance Principle of MLE's)

If $\hat{\underset{\sim}{\theta}}$ is the MLE of $\underset{\sim}{\theta}$, then for any function τ of $\underset{\sim}{\theta}$, $\tau(\hat{\underset{\sim}{\theta}})$ is the MLE of $\tau(\underset{\sim}{\theta})$.

11] $\log(x) = \ln(x)$. If $L(\theta) > 0 \ \forall \ \theta \in \Theta$ and $\hat{\theta}$ maximizes $\log L(\theta)$, then $\hat{\theta}$ maximizes $L(\theta)$.

12] Suppose that θ is one dimensional, $L(\theta) > 0$ is continuous on Θ, and $\Theta = [a,b]$ is closed and bounded. Then there exists a $\hat{\theta} \in \Theta$ that maximizes $\log L(\theta)$. Suppose $\frac{d}{d\theta} \log L(\theta)$ exists everywhere on (a,b) and that $\frac{d}{d\theta} \log L(\theta) \stackrel{set}{=} 0$ (or $\frac{d}{d\theta} L(\theta) \stackrel{set}{=} 0$) has