a unique solution $\underbrace{\theta_0 \in \Theta}_{\text{critical point}}$, then

$$\hat{\theta}(\underline{Y}) = \begin{cases} \theta_0(\underline{Y}) \\ a \\ b \end{cases} \quad \text{depending on which maximizes } \log L(\theta)$$

For general Θ, eg $\Theta = (-\infty, \infty)$, we still need to check the boundaries.

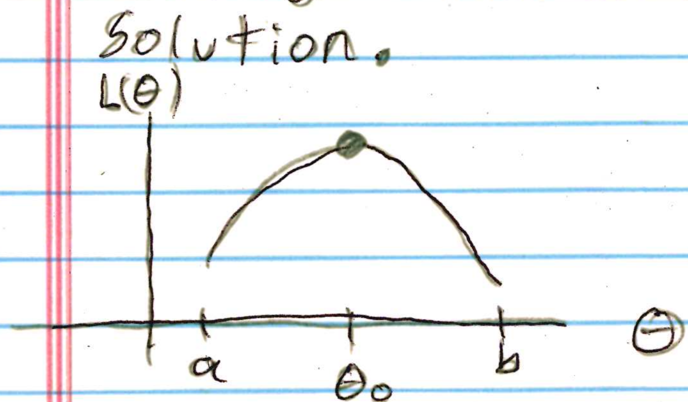
13) Know If $\frac{d}{d\theta} \log L(\theta)$ exists on (a,b) where $\Theta = [a,b]$ and

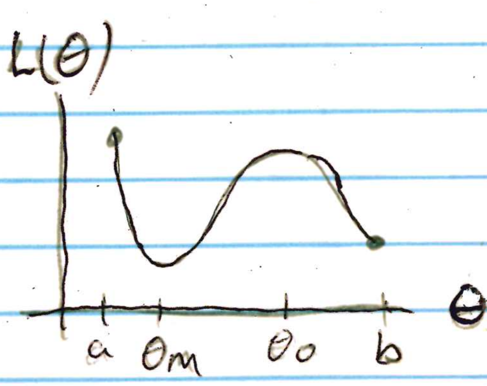
if $\frac{d}{d\theta} \log L(\theta)\Big|_{\theta=\theta_0} = 0$ and $\frac{d^2}{d\theta^2} \log L(\theta)\Big|_{\theta=\theta_0} < 0$

$\left(\text{or } \frac{d^2}{d\theta^2} L(\theta)\Big|_{\theta=\theta_0} < 0\right)$,

then θ_0 is a local max.

If θ_0 was the unique solution in Θ to $\frac{d}{d\theta} \log L(\theta) \overset{set}{=} 0$, then θ_0 is the global max since if a or b is the global max, then there would be a local min, say θ_m, and $\frac{d}{d\theta} \log L(\theta)\Big|_{\theta_m} = 0$, and θ_0 would not be the unique solution.





Note: Always check the boundaries of Θ because it is easy to make a mistake.

14] $\boxed{\text{Common Final Problem}}$ Given

$Y_1, \ldots, Y_n$ are iid with pdf $f(y|\underset{\sim}{\theta})$
or prob function $p(y|\underset{\sim}{\theta})$,

find the MLE $\hat{\underset{\sim}{\theta}}(\underset{\sim}{Y})$.

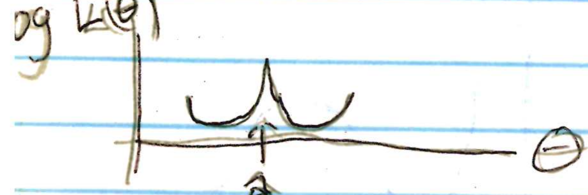
Techniques for finding potential candidates

$\ast$ i) Differentiate the log likelihood $\log L(\theta)$.
ii) Directly maximize the likelihood $L(\theta)$.
$\ast$ iii) If $\hat{\theta}$ is the MLE of θ, then $\tau(\hat{\theta})$ is the MLE of $\tau(\theta)$
iv) Differentiate $L(\theta)$.

v) If $\Theta = [a, b]$, solve

$$(\ast) \quad \frac{d}{d\theta} \log L(\theta) \overset{set}{=} 0 \quad \left(\text{or} \quad \frac{d}{d\theta} L(\theta) \overset{set}{=} 0 \qquad (\ast\ast) \right).$$

The solutions to $(\ast)$ (or $(\ast\ast)$) are potential candidates for the MLE. Other candidates are the boundary points a and b of Θ and $\theta^* \in \Theta$ where the derivative does not exist.



15] The candidates could be local max's or local mins.
Want the global max. Point 13] is a good way to show $\hat{\theta}$ is a global max.

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16] Let $Y_1, \dots, Y_n$ be the sample.

The order statistics are $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ where $Y_{(1)} \le Y_{(2)} \le Y_{(3)} \le \dots \le Y_{(n-1)} \le Y_{(n)}$.

$Y_{(1)}$ is the min : $Y_{(1)} = \min(Y_1, \dots, Y_n)$

$Y_{(n)}$ is the max : $Y_{(n)} = \max(Y_1, \dots, Y_n)$

ex] data 1, 7, 19, 24, 4, 13

order data 1, 4, 7, 13, 19, 24

$Y_{(1)}$ $Y_{(2)}$ $Y_{(3)}$ $Y_{(4)}$ $Y_{(5)}$ $Y_{(6)}$

The min = smallest value = $Y_{(1)} = 1$.

The max = largest value = $Y_{(6)} = 24$.

$n = 6$

ex] $Y_1, \dots, Y_n$ are iid normal μ, σ^2 where $\sigma^2 > 0$ is known. Find the MLE of μ.

Soln $f_{Y_i}(y_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\frac{-1}{2\sigma^2}(y_i - \mu)^2\right]$.

So $L(\mu) = \prod_{i=1}^{n} f_{Y_i}(y_i) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left[\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right]$

and $\log L(\mu) = \log\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2$.

$\frac{d \log L}{d\mu}(\mu) = 0 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} 2(y_i - \mu)(-1) = \frac{1}{\sigma^2} \sum (y_i - \mu)$

or $\frac{1}{\sigma^2}\left(\sum Y_i - n\mu\right) \stackrel{set}{=} 0$

$$n\mu = \sum_{i=1}^{n} Y_i$$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y}$$

$\hat{\mu}$ was the unique solution to $\frac{d}{d\mu}\log L(\mu) = 0$

and $\frac{d^2 \log L(\mu)}{d\mu^2} = \frac{d}{d\mu}\left(\frac{1}{\sigma^2}\sum Y_i - \frac{n\mu}{\sigma^2}\right) =$

$-\frac{n}{\sigma^2} < 0$. Hence $\hat{\mu}$ is

the global max and $\boxed{\hat{\mu}(\underset{\sim}{Y}) = \bar{Y}}$.

Alternatively,

$$\log L(\mu) = \underbrace{c}_{\text{constant wrt } \mu} - \frac{1}{2\sigma^2}\left(\sum Y_i^2 - 2\mu\sum Y_i + n\mu^2\right)$$

so $\frac{d}{d\mu}\log L(\mu) = \frac{-1}{2\sigma^2}\left(-2\sum Y_i + 2n\mu\right) \stackrel{set}{=} 0$

or $2n\mu = 2\sum Y_i$, or $\hat{\mu} = \frac{\sum Y_i}{n}$

as before.

17) * Fact: For any number a,

$$\sum_{i=1}^{n}(y_i - a)^2 \geq \sum_{i=1}^{n}(y_i - \bar{y})^2$$

with equality iff $a = \bar{y}$.

18] Let $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)$.

Solve (*) $\frac{\partial}{\partial \theta_i} \log L(\underset{\sim}{\theta}) \overset{set}{=} 0$

(or (**) $\frac{\partial}{\partial \theta_i} L(\underset{\sim}{\theta}) \overset{set}{=} 0$) for $i = 1, \ldots, k$.

The solutions to (*) (or (**)) are potential candidates for the MLE.

Other candidates are on the boundary of Θ and where the partial derivatives do not exist.

The solutions of (*) and (**) could be local min's or max's.

19] If $k > 1$, we will assume in this class that at least one of the solutions to $\frac{\partial}{\partial \theta_i} \log L(\underset{\sim}{\theta}) \overset{set}{=} 0$ is the MLE. (I won't give you any problems where this assumption is false. In general, you need to check whether the critical points are local max's and then if they are global max's, but this checking takes too long.)

ex] Let $Y_1, \ldots, Y_n$ be iid $N(\mu, \sigma^2)$ where both μ and σ^2 are unknown.

$$L(\mu, \sigma^2 \mid \underset{\sim}{y}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\frac{-1}{2\sigma^2}(y_i - \mu)^2\right]$$

$$= \left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{1}{\sigma^2}\right)^{n/2} \exp\left[\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right]$$

$$\log L(\mu, \sigma^2) = \log\left[\left(\frac{1}{2\pi}\right)^{n/2}\right] - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2.$$

It may be convenient to let $\sigma^2 = \tau$.

then $$\log L(\mu, \tau) = \log\left[\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\right] - \frac{n}{2} \log \tau - \frac{1}{2\tau} \sum_{i=1}^{n} (y_i - \mu)^2.$$

Now $$\frac{\partial}{\partial \mu} \log L(\mu, \sigma^2) = \frac{1}{\sigma^2} \Sigma (y_i - \mu)$$

and $$\frac{\partial}{\partial \sigma^2} \log L(\mu, \sigma^2) = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (y_i - \mu)^2$$

Since $$\frac{\partial}{\partial \tau} \log L(\mu, \tau) = -\frac{n}{2}\frac{1}{\tau} - \left(\frac{-1}{2\tau^2}\right) \Sigma (y_i - \mu)^2$$

$$= -\frac{n}{2}\frac{1}{\tau} + \frac{1}{2\tau^2} \sum_{i=1}^{n} (y_i - \mu)^2.$$

Solve $$\frac{1}{\sigma^2} \Sigma (y_i - \mu) \overset{\text{set}}{=} 0 \qquad \text{(i)}$$

$$-\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (y_i - \mu)^2 \overset{\text{set}}{=} 0 \qquad \text{(ii)}$$

Get $\Sigma Y_i = n\,\mu$ or $\hat{\mu} = \frac{\Sigma Y_i}{n} = \bar{Y}$ . . 483 69

plug $\hat{\mu}$ into ii) to get

$$\sigma^4 \left(-\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \Sigma (Y_i - \bar{Y})^2 \overset{set}{=} 0 \right)$$

or $$\frac{n}{2}\sigma^2 = \frac{1}{2}\Sigma (Y_i - \bar{Y})^2$$

or $$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (Y_i - \bar{Y})^2}{n}.$$

so the MLE is $(\hat{\mu}, \hat{\sigma}^2) = (\bar{Y}, \frac{1}{n}\Sigma (Y_i - \bar{Y})^2)$.

Skip ↓

Note: We can actually verify that $(\hat{\mu}, \hat{\sigma}^2)$ is the global max.

$$L(\mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \underbrace{e^{-\frac{1}{2\sigma^2}\Sigma (Y_i - \mu)^2}}$$

maximize L by maximizing $e^{-\frac{1}{2\sigma^2}\Sigma (Y_i - \mu)^2}$

or by minimizing $\frac{1}{2\sigma^2}\Sigma (Y_i - \mu)^2$,

or, for $\sigma^2 > 0$, minimize

$$\Sigma (Y_i - \mu)^2.$$

Hence $\hat{\mu} = \bar{Y}$ regardless of $\sigma^2 > 0$.

That is $\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\Sigma(y_i-\bar{y})^2} \geq \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\Sigma(y_i-\mu)^2}$

for any value of σ^2.

Now the problem is one dimensional:

find the global max of

$$\tilde{L}(\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\Sigma(Y_i-\bar{Y})^2}$$

or of $\log \tilde{L}(\sigma^2) = \log\left(\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\right) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Sigma(Y_i-\bar{Y})^2$

But $\frac{d}{d\sigma^2}\log\tilde{L}(\sigma^2) = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\Sigma(Y_i-\bar{Y})^2 \overset{set}{=} 0$

or $\frac{n}{2}\sigma^2 = \frac{1}{2}\Sigma(Y_i-\bar{Y})^2$

or $\hat{\sigma}^2 = \frac{1}{n}\Sigma(Y_i-\bar{Y})^2$.

$\hat{\sigma}^2$ is a unique critical point

and $\tilde{L}(0) = 0$ ($e^{-\frac{1}{0}}$ dominates)

$\tilde{L}(\infty) = 0$

↑ Skip

so $\hat{\sigma}^2$ is the global max.

ex] Direct maximization:

$Y_1, \ldots, Y_n$ are iid $N(\mu, 1)$

want $-\frac{1}{2}\Sigma(Y_i-\mu)^2$ as close to 0 as possible

$$L(\mu) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(Y_i-\mu)^2\right] = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\exp\left[\underbrace{-\frac{1}{2}\Sigma(Y_i-\mu)^2}\right].$$

Maximize $L(\mu)$ by maximizing $\exp\left[-\frac{1}{2}\Sigma(Y_i-\mu)^2\right]$

or by minimizing $\Sigma(y_i - \mu)^2$

but $\Sigma(y_i-\mu)^2 \geq \Sigma(y_i - \bar{y})^2$

so $\hat{\mu} = \bar{Y}$ is the MLE

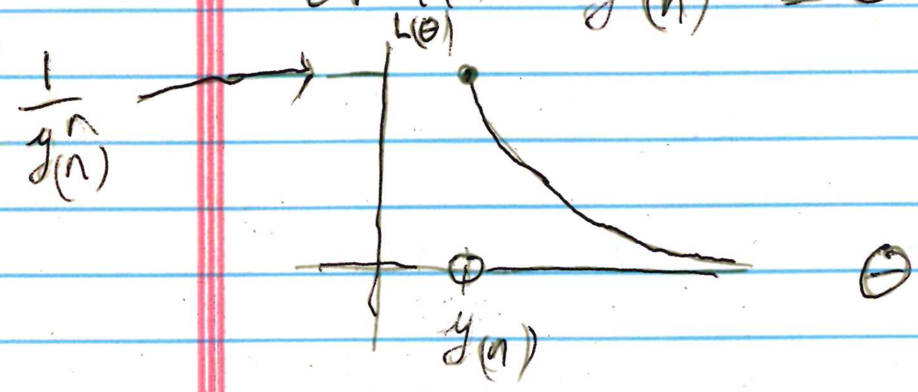
ex) $Y_1, \ldots, Y_n$ iid Uniform $(0,\theta)$, $\theta \geq 0$

so $\Theta = [0, \infty)$.

$f(y|\theta) = \frac{1}{\theta}$ if $0 \leq y \leq \theta$.

so $L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \frac{1}{\theta^n}$ if $0 \leq y_1, \ldots, y_n \leq \theta$

or if $y_{(n)} \leq \theta$



so $\hat{\theta} = Y_{(n)} = \max(Y_1, \ldots, Y_n)$

by direct maximization.

The MLE of $\sin(\theta)$ is $\sin(\hat{\theta}) = \sin(Y_{(n)})$.

see HW 2 1. c

Note: let $I(x \in A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$

Then $f(y_i|\theta) = \frac{1}{\theta} I(0 \leq y_i \leq \theta)$ and $L(\theta) = \frac{1}{\theta^n} I(y_{(n)} \leq$

Do MLE review e) and f)

§9.3 20] p450 * An estimator $\hat{\theta}_n$ is a <u>consistent estimator of θ</u> if for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P\left(|\hat{\theta}_n - \theta| \le \varepsilon\right) = 1$$

(equivalently if $\lim_{n \to \infty} P\left(|\hat{\theta}_n - \theta| > \varepsilon\right) = 0$).

21] * If $\hat{\theta}_n$ is an unbiased estimator of θ ($E\hat{\theta}_n = \theta$ so $\text{Bias}(\hat{\theta}_n) = E\hat{\theta}_n - \theta = 0$), then $\hat{\theta}_n$ is a consistent estimator of θ if $\lim_{n \to \infty} V(\hat{\theta}_n) = 0$.

ex] p451 * Law of Large numbers

$Y_1, \ldots, Y_n$ iid with $EY_i = \mu$ and $V(Y_i) = \sigma^2$.

Then $E\bar{Y} = \mu$, $V(\bar{Y}) = \frac{\sigma^2}{n} \to 0$.

So $\bar{Y}$ is a consistent estimator of μ. applications { Gambling: few large bets min's losses
Stocks: buy small amounts of many stocks so portfolio behaves like ave

M483 MLE

$\neg 0^{3/4}$

$$f(y) = \mu \exp[(\mu-1)\log(y)], \quad 0<y<1, \quad \mu>0.$$

$$L(\mu) = \mu^n \exp[(\mu-1)\Sigma \log(y_i)]$$

$$\log L(\mu) = n\log\mu + (\mu-1)\Sigma\log(y_i)$$

$$\frac{d\log L(\mu)}{d\mu} = \frac{n}{\mu} + \Sigma\log(y_i) \stackrel{set}{=} 0$$

or $n = -\mu\Sigma\log y_i$

or $\hat{\mu} = \dfrac{-n}{\sum_{i=1}^{n}\log(y_i)}$, <u>unique</u>

$$\frac{d^2\log L(\mu)}{d\mu^2} = -\frac{n}{\mu^2} < 0$$

so $\hat{\mu}$ is the MLE

$f(x) = \theta \exp[-(\theta+1)\log(1+x)], \quad \theta > 0, x > 0$

$L(\theta) = \theta^n \exp\left[-(\theta+1)\sum \log(1+x_i)\right]$

$\log L(\theta) = n\log(\theta) - (\theta+1)\sum \log(1+x_i)$

$$\frac{d\log L(\theta)}{d\theta} = \frac{n}{\theta} - \sum \log(1+x_i) \overset{set}{=} 0$$

$$\hat{\theta} = \frac{n}{\sum \log(1+x_i)}, \quad \underline{\text{unique}}$$

$$\frac{d^2\log L(\theta)}{d\theta^2} = \frac{-n}{\theta^2} < 0$$ so $\hat{\theta}$ is the MLE

$f(x) = \frac{\theta}{2}\exp\left[-(\theta+1)\log(1+|x_i|)\right], \quad x \in \mathbb{R}, \theta > 0$

$L(\theta) = \frac{1}{2^n}\theta^n \exp\left[-(\theta+1)\sum \log(1+|x_i|)\right]$

$\log L(\theta) = \log\frac{1}{2^n} + n\log(\theta) - (\theta+1)\sum \log(1+|x_i|)$

$$\frac{d\log L(\theta)}{d\theta} = \frac{n}{\theta} - \sum \log(1+|x_i|) \overset{set}{=} 0$$

$$\hat{\theta} = \frac{n}{\sum \log(1+|x_i|)}, \quad \underline{\text{unique}}$$

$$\frac{d^2\log L(\theta)}{d\theta^2} = \frac{-n}{\theta^2} < 0$$ so $\hat{\theta}$ is the MLE

22] *If $\lim_{n \to \infty} E\hat{\theta}_n = \theta$ and $\lim_{n \to \infty} V(\hat{\theta}_n) = 0$.

or if $\lim_{n \to \infty} MSE(\hat{\theta}_n) = 0$, then $\hat{\theta}_n$ is a consistent estimator of θ.

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23] *Typically $T = T_n = \hat{\theta}_n = c \sum_{i=1}^n Y_i$ where $Y_1, ..., Y_n$ are iid with $EY_i = \mu$ and $V(Y_i) = \sigma^2$.

Then $E(\hat{\theta}_n) = c \sum_{i=1}^n EY_i = nc\mu$

and $V(\hat{\theta}_n) = c^2 V\left(\sum_{i=1}^n Y_i\right) = c^2 \sum_{i=1}^n V(Y_i)$

$= nc^2\sigma^2$,

In particular, if $\hat{\theta}_n = k\bar{Y} = \frac{k}{n} \sum_{i=1}^n Y_i$,

then $E k\bar{Y} = k\mu$ $\leftarrow$ $\left(c = \frac{k}{n}\right)$

$V k\bar{Y} = k^2 \frac{\sigma^2}{n}$

24] Common Final Problem $\hat{\theta}_n = c \sum_{i=1}^n Y_i$ or $k\bar{Y}$. Data $Y_1, ..., Y_n$ are iid. Find the MSE of $\hat{\theta}_n$ as a function of c and n, and choose c so that $\hat{\theta}_n$ is a consistent estimator of θ.

ex] $Y_1, \ldots, Y_n$ are iid with $E\,Y_i = \frac{\theta}{2} = \mu$, $\sigma^2 = V\,Y_i = \frac{\theta^2}{12}$. Let $T_n = c\bar{Y}$ estimate θ. Find the MSE of T_n as a function of n and c and choose c so that T_n is consistent for θ.

Soln] $MSE = V(T_n) + [B(T_n)]^2$

$$B(T_n) = E\,T_n - \theta = c\mu - \theta = \frac{c\theta}{2} - \theta$$

and $V(T_n) = V(c\bar{Y}) = c^2 \frac{\sigma^2}{n} = \frac{c^2\theta^2}{12n}$.

$$\text{So } MSE = \frac{c^2\theta^2}{12n} + \left(\frac{c\theta}{2} - \theta\right)^2$$

$$\lim_{n\to\infty} MSE(T_n) = 0 + \left(\frac{c\theta}{2} - \theta\right)^2 = 0$$

if $c = 2$.

so $T_n = 2\bar{Y}$ is a consistent estimator for θ.

Note that $E\,2\bar{Y} = 2\mu = 2\frac{\theta}{2} = \theta$ unbiased, and $V(2\bar{Y}) = 4\frac{\sigma^2}{n} = \frac{4\theta^2}{12n} \to 0$, so $2\bar{Y}$ is consistent.