common problem

ex] Easier way to do § 2.9 ex

Fair die is rolled 3 times

Y counts the number of 5's.

Then $Y \sim \text{bin}(n=3, p=\frac{1}{6})$.

| $y$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(y)$ | .5787 | .3472 | .0694 | .0046 |

sum = .9999

*a) $EY = np = 3\frac{1}{6} = 0.5$

*b) $VY = np(1-p) = 3\frac{1}{6}\frac{5}{6} = 5/12 = .41667$

i) $P(\underline{\text{none}}$ of the rolls are 5's$) = P(Y=0) = \binom{3}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3$

$= \left(\frac{5}{6}\right)^3 = p(0)$

ii) $P(\underline{\text{at least one}}$ roll is a 5$) = P(Y \geq 1)$

$= p(1) + p(2) + p(3) = 1 - \left(\frac{5}{6}\right)^3 = 1 - p(0)$

iii) $P(\underline{\text{all}}$ 3 rolls are 5's$) = P(Y=3) = \binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0$

$= \left(\frac{1}{6}\right)^3 = p(n)$

iv) $P($not all 3 rolls are 5's$) = P(Y \neq 3) =$

$1 - P(Y=3) = p(0) + p(1) + p(2) = 1 - p(3)$

$= 1 - \left(\frac{1}{6}\right)^3 = 1 - p(n)$.

v) $P($at most one roll is a 5$) = P(Y \leq 1)$

$= p(0) + p(1) = \binom{3}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 + \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2$

$= \left(\frac{5}{6}\right)^3 + 3\frac{1}{6}\left(\frac{5}{6}\right)^2 \approx .5787 + .3472$

$= .9259$.

vi) $P($at least two rolls are 5's$) = P(Y \geq 2)$

$= p(2) + p(3) = 1 - p(0) - p(1)$

$= 1 - .9259 = 0.0741$.

14.5

23] p108 common error $q = 1-p$ is given in story problem and student uses $Y \sim bin(n, q)$ instead of $Y \sim bin(n, p)$.

see HW5 $H_0$.

skip ex 3.10.

END EXAM 1 MATERIAL

§3.5 24] p115 If $W \sim bin(n,p)$, then W counts the number of successes in n trials. If Y is the trial at which the 1st success occurs, then Y has a Geometric (p) distribution.

$$P(Y=y) = P(\text{1st } y-1 \text{ trials are F, } y\text{th S})$$
$$= P(FF \cdots FS) = (1-p)^{y-1}p, \quad y = 1, 2, \ldots$$

25] A RV Y is geometric (p) if

$P(Y=y) = p(y) = (1-p)^{y-1} p$, $y = 1, 2, 3, \ldots$
$0 \le p \le 1$.

26] p116 If $Y \sim geom(p)$, then $EY = \frac{1}{p}$ and $V(Y) = \frac{1-p}{p^2}$.

Skip §3.6 and §3.7

§3.8 p131 A Poisson($\lambda$) RV Y is used to model the total # of occurrences of some phenomenon in a fixed period of time or a fixed region of space where $\lambda$ is the average value of Y. eg # telephone calls in a 5 minute interval
# radioactive particles that hit a target in 3 hours.

27] KNOW p132 A RV $Y$ has a Poisson($\lambda$) distribution if $p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$, $y = 0,1,2,\ldots$ $\lambda > 0$.

28] p131 Suppose that $W \sim \text{bin}(n,p)$ where $n$ is large and $p$ small so that $np < 7$. Then the pois ($\lambda = np$) RV $Y$ is a good approx to $W$ in that

$$p_W(w) = \binom{n}{w} p^w (1-p)^{n-w} \approx \frac{\lambda^w e^{-\lambda}}{w!} = p_Y(w).$$

29] $$\sum_{y=0}^{\infty} p(y) = e^{-\lambda} \underbrace{\sum_{y=0}^{\infty} \frac{\lambda^y}{y!}}_{\text{Taylor (or maclaurin) series expansion for } e^x,\ x=\lambda \text{ see p 836}} = e^{-\lambda} e^{\lambda} = 1$$

30) Know If $Y$ is Poisson($\lambda$), then $E(Y) = \lambda$ and $V(Y) = \lambda$.

proof] $$E(Y) = \sum_{y=0}^{\infty} y \frac{\lambda^y e^{-\lambda}}{y!} \overset{\text{trick}}{=} \lambda \sum_{y=1}^{\infty} \frac{\lambda^{y-1} e^{-\lambda}}{(y-1)!}$$

trick: Pull out a $\lambda$ and $\frac{y}{y!} = \frac{1}{(y-1)!}$, $y=0$ term drops out

Take $z = y-1$, $y=\infty \to z=\infty$, $y=1 \to z=0$

omit if running late

$$E(Y) \overset{\text{trick}}{=} \lambda \underbrace{\sum_{z=0}^{\infty} \frac{\lambda^z e^{-\lambda}}{z!}}_{\sum p(z) = 1} = \lambda$$

$$E[Y(Y-1)] \overset{\text{trick}}{=} \sum_{y=0}^{\infty} y(y-1) \frac{\lambda^y e^{-\lambda}}{y!} = \lambda^2 \sum_{y=2}^{\infty} \frac{\lambda^{y-2} e^{-\lambda}}{(y-2)!}$$

$z = y-2$ $\quad y = \infty \to z = \infty$ $\quad y = 2 \to z = 0$

$$\overset{\downarrow}{=} \lambda^2 \sum_{z=0}^{\infty} \frac{\lambda^z e^{-\lambda}}{z!} = \lambda^2.$$

So. $E[Y^2 - Y] = \lambda^2$ and $E Y^2 = \lambda^2 + \lambda$.

$\therefore$ $V(Y) = E Y^2 - [E(Y)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

common E2 problem

ex] Suppose that $Y$ counts the number (15.5) of radioactive particles that hit a target in two minutes and that $Y$ is Poisson with $E(Y) = 6$. a) Find the prob that the # of strikes in a future 2 minute period is greater than or equal to 10.

Soln] $\lambda = E(Y) = 6$ want $P(Y \geq 10)$.

Table 3 on P 845 gives $P(Y \leq a)$

Now $P(Y \geq 10) = 1 - P(Y \leq 9)$

| $\lambda \backslash a$ | 9 |
|---|---|
| 6 | 0.916 |

So $P(Y \geq 10) = 1 - .916 = 0.084$

* b) Find $P(Y=4) = p(4) = \dfrac{\lambda^4 e^{-\lambda}}{4!}$

$= \dfrac{6^4 e^{-6}}{4!} = 0.13385$

or $P(Y=4) = P(Y \leq 4) - P(Y \leq 3)$

table3

| $\lambda \backslash a$ | 3 | 4 |
|---|---|---|
| 6 | 0.151 | 0.285 |

$P(Y=4) = .285 - .151 = .134$

§3.9 P138 31] * The kth moment of $Y$ (about the origin) is $E(Y^k)$ $(= \mu_k')$.

32] The kth central moment of $Y$ is $E[(Y - E(Y))^k]$ $(= \mu_k)$.

Note $k=2$ gives $V(Y)$.

33] p139 Know] The moment generating function (mgf) for a RV $Y$ is $m(t) = E(e^{tY})$. The mgf exists if $m(t)$ is finite for $|t| \le b$ for some $b > 0$.

34] If $Y$ is discrete, $m(t) = \sum_{y : p(y) > 0} e^{ty} p(y)$.

35] Common Problem Given a table for $p(y)$, find $m(t) = E(e^{tY})$.

ex]

| $y$ | $-1$ | $+1$ |
|---|---|---|
| $p(y)$ | 0.5 | 0.5 |

$m(t) = E(e^{tY}) = \sum_y e^{ty} p(y) = e^{-t} 0.5 + e^{t} 0.5$

$= \frac{1}{2}\left[e^{-t} + e^{t}\right]$.

36] p139 Know Let $Y$ be a RV with mgf $m(t)$. Then $E[Y^k] = \left.\frac{d^k m(t)}{dt^k}\right]_{t=0} \equiv m^{(k)}(0)$.

37] Common Problem Given $m(t)$, use point 36] to find $E(Y)$ and $E(Y^2)$.

38] $\star$ p140 The mgf of a Poisson($\lambda$) RV is

$$m(t) = e^{\lambda(e^t - 1)}$$

proof] $m(t) = E e^{tY} = \sum_{y=0}^{\infty} e^{ty} \frac{\lambda^y e^{-\lambda}}{y!} = e^{-\lambda} \sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!}$

(taylor series p836 with $x = \lambda e^t$)

$\overset{\downarrow}{=} e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$ $\forall t$.

165

ex] Suppose $m(t) = e^{\lambda(e^t - 1)}$.

Find $E(Y)$ and $V(Y)$ using the mgf.

Soln $\frac{d}{dt} m(t) = \underbrace{e^{\lambda(e^t-1)}}_{g'(f(t))} \underbrace{\lambda e^t}_{f'(t)}$ ← chain rule

chainrule

(use product rule for $m''(t)$)

$[g(f(x))]' = g'(f(x)) f'(x)$

So $m'(0) = e^{\lambda(e^0-1)} \lambda e^0 = \lambda = E(Y)$

Now $\frac{d^2}{dt^2} m(t) = \underbrace{e^{\lambda(e^t-1)} \lambda e^t}_{f'(t)} \underbrace{\lambda e^t}_{g(t)} + \underbrace{e^{\lambda(e^t-1)}}_{f(t)} \underbrace{\lambda e^t}_{g'(t)}$.

product rule on $\frac{d}{dt} m(t)$

$[f(t)\, g(t)]' = f'(t) g(t) + f(t) g'(t)$

So $m^{(2)}(0) = e^{\lambda(1-1)} \lambda e^0 \lambda e^0 + e^{\lambda(1-1)} \lambda e^0$

$= \lambda^2 + \lambda$.

So $\sigma^2 = E(Y^2) - (E(Y))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

as before.

39]* The mgf of a bin $(n, p)$ RV is

$m(t) = [p e^t + (1-p)]^n \quad \forall t$.

proof $m(t) = \sum_{i=0}^{n} e^{ti} \binom{n}{i} p^i (1-p)^{n-i}$

$= \sum_{i=0}^{n} \binom{n}{i} (pe^t)^i (1-p)^{n-i}$

$= [pe^t + (1-p)]^n$

(binomial theorem p 45 with $x = 1-p$, $y = pe^t$)

40] p141 Suppose that $W$ and $Y$ are 2 RV's with $m_W(t) = m_Y(t)$ $\forall |t| < b$ for some $b > 0$. Then $W$ and $Y$ have the same prob. dist.

41] p146 * Tchebysheff's Theorem. 483 17

Let $Y$ be a RV with $E(Y) = \mu$ and $V(Y) = \sigma^2$.

If $k > 0$, then $P(|Y-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$,

or $P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}$.

Note: $\sigma$ is a measure of spread

| | Tchebysheff | Empirical ~~68-95-99.7~~ 68-95-99.7 |
|---|---|---|
| $P(\lvert Y-\mu\rvert < 2\sigma) \geq$ | $.75$ | $\approx .95$ |
| $P(\lvert Y-\mu\rvert < 3\sigma) \geq$ | $8/9 \approx .889$ | $\approx .997$ |
| | any dist with $V(Y) < \infty$ | bell shaped |

ch4 1] p158 know Let $Y$ be any RV.

The distribution function of $Y$ is $F(y) = P(Y \leq y)$ for $-\infty < y < \infty$.

Note] If $Y$ is discrete with probability function $p(x)$, then
$$F(y) = \sum_{\{x :\, p(x) > 0 \text{ and } x \leq y\}} p(x).$$

2] * p160 If $F(y)$ is a dist fn, then

df1) $F(-\infty) = \lim_{y \to -\infty} F(y) = 0$

df2) $F(\infty) = \lim_{y \to \infty} F(y) = 1$

df3) $F(y)$ is a nondecreasing function of $y$: if $y_1 < y_2$, then $F(y_1) \leq F(y_2)$.

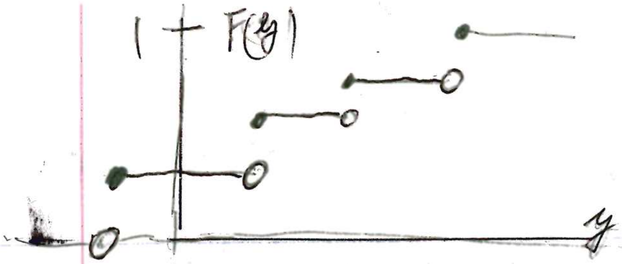
not in book

df4) $F(y)$ is right continuous: $\lim_{y \downarrow y_0} F(y) = F(y_0)$ for any $y_0$.

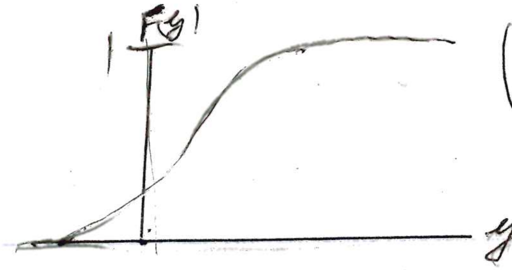
Note if $A = \{Y \leq y_1\}$ and $B = \{Y \leq y_2\}$ and $y_1 < y_2$, then $A \subseteq B$.

Also $\{Y \leq y_1\} = \{s \in S \mid Y(s) \leq y_1\}$.

17.5




F(y) is a step fn if Y is discrete




F(y) for Y continuous

p160
3) Know The RV Y is continuous if the df F(y) is continuous for $-\infty < y < \infty$.

4) p161 Know Let F(y) be the df of continuous Y and suppose $f(y) = \frac{dF(y)}{dy}$ wherever the derivative exists. Then f(y) is called the probability density function (pdf) of Y. (Assumption $\frac{dF}{dy}$ exists and is continuous except for at most a finite # of points in any finite interval.)

5) Know $F(y) = \int_{-\infty}^{y} f(t)\,dt$

6) p 162 If f(y) is a pdf, then
pdf 1) $f(y) \geq 0 \quad \forall\ y \in (-\infty, \infty) = \mathbb{R}$
pdf 2) $\int_{-\infty}^{\infty} f(y)\,dy = 1$.

7) * p165 If Y has pdf f(y) and $a \leq b$, then $P(a \leq Y \leq b) = \int_a^b f(y)\,dy = F(b) - F(a)$.

$\leq$ means $<$ or $\leq$

For a discrete RV W, $P(a < W \leq b) = F(b) - F(a)$ since $F(a) = P(W \leq a)$. If Y is a continuous RV, then $F(a) = P[Y \leq a] = P[Y < a]$ since $P(Y = a) = 0$.

8) Common problem a) Find f(y) from F(y).
b) Find F(y) from f(y).
c) Told $f(y) = c\, g(y)$. Find c.