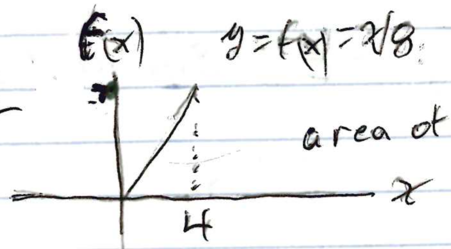
ex) X RV with $f(x) = \begin{cases} cx & 0<x<4 \\ 0 & \text{else} \end{cases}$

$$\int_0^4 cx\,dx = c\frac{x^2}{2}\Big|_0^4 = 8c = 1$$

so $c = \frac{1}{8}$. so $f(x) = \begin{cases} x/8 & 0<x<4 \\ 0 & \text{else} \end{cases}$

$\frac{4}{8} = \frac{1}{2}$



area of triangle $= 1 = \frac{1}{2}\left(\frac{1}{2}\right)4$

$= \frac{1}{2}(\text{base})(\text{ht})$
$= \frac{1}{2}(\text{ht})(\text{base})$

Now $F(y) = \begin{cases} 0, & y \le 0 \\ 1, & y \ge 4 \\ \frac{y^2}{16}, & 0<y<4 \end{cases}$

$\int_0^y \frac{t}{8}\,dt = \frac{t^2}{2(8)}\Big|_0^y =$

Check: Note that $\frac{d}{dy}F(y) = \begin{cases} 0, & y \le 0 \\ 0, & y \ge 4 \\ 2\frac{y}{16} = \frac{y}{8}, & 0<y<4. \end{cases}$

$$P(1 \le X \le 2) = \int_1^2 \frac{x}{8}\,dx = \frac{x^2}{16}\Big|_1^2 = \frac{4}{16} - \frac{1}{16} = \frac{3}{16}.$$

$$P(X \le 1 \mid X \le 2) = \frac{P(\{X \le 1\} \cap \{X \le 2\})}{P(X \le 2)}$$

$$= \frac{P(X \le 1)}{P(X \le 2)} = \frac{F(1)}{F(2)} = \frac{1/16}{4/16} = \frac{1}{4}.$$

§4.3
9] Know If Y has pdf $f(y)$, then the expected value of Y is $E(Y) = \int_{-\infty}^{\infty} y\,f(y)\,dy$,

and $E(g(Y)) = \int_{-\infty}^{\infty} g(y)\,f(y)\,dy$

provided the integrals exist.

common error! get $EY = 1$ because y is forgotten.

18.5

p95 &

10] p171 Properties of expectation

Let c be a constant and $g, g_1, \dots, g_k$ functions of (continuous or discrete) RV Y.

Then E1) $E(c) = c$

E2) $E(c\,g(Y)) = c\,E(g(Y))$

E3) $E[g_1(Y) + \dots + g_k(Y)] = E(g_1(Y)) + \dots + E(g_k(Y))$

11] p171 know If Y has pdf $f(y)$,

$$V(Y) = \int_{-\infty}^{\infty} (y - E(Y))^2 f(y)\,dy.$$

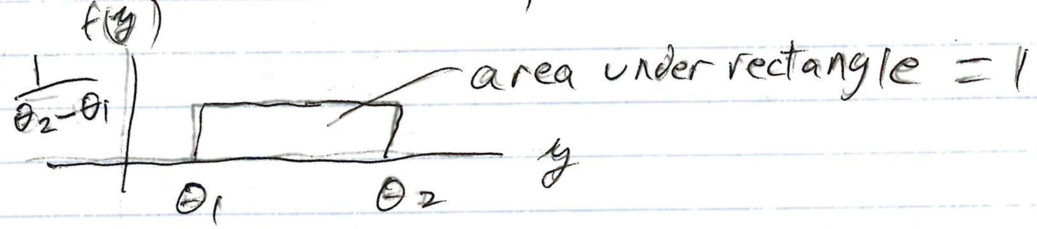
12] know p171 Short cut formula $V(Y) = E(Y^2) - [E(Y)]^2$

where $E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy$ for Y contin.

§4.4

13] p174 know If $\theta_1 < \theta_2$, Y is uniform (θ_1, θ_2) if the pdf of Y

is $f(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2 \\ 0, & \text{else} \end{cases}$



14] p176 know $E(Y) = \frac{\theta_1 + \theta_2}{2}$, $V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}$

proof of V(Y); $EY^2 = \int_{\theta_1}^{\theta_2} \frac{x^2}{\theta_2 - \theta_1}\,dx = \frac{x^3}{3(\theta_2 - \theta_1)}\Big|_{\theta_1}^{\theta_2}$

$= \frac{\theta_2^3 - \theta_1^3}{3(\theta_2 - \theta_1)}$

$$\begin{array}{r|l} & \theta_2^2 + \theta_2\theta_1 + \theta_1^2 \\ \hline \theta_2 - \theta_1 & \theta_2^3 - \theta_1^2 \\ & \theta_2^3 - \theta_2^2\theta_1 \\ \hline & \theta_2^2\theta_1 - \theta_1^2 \\ & \theta_2^2\theta_1 - \theta_2\theta_1^2 \\ \hline & \theta_2\theta_1^2 - \theta_1^2 \\ & \theta_2\theta_1^2 - \theta_1^3 \end{array}$$

so $V(Y) = EY^2 - (EY)^2 = \frac{\theta_2^2 + \theta_2\theta_1 + \theta_1^2}{3} - \left(\frac{\theta_1 + \theta_2}{2}\right)^2$

or $$V(Y) = \frac{\theta_2^2 + \theta_2\theta_1 + \theta_1^2}{3} - \frac{\theta_1^2 + 2\theta_1\theta_2 + \theta_2^2}{4}.$$

$$= \frac{4\theta_2^2 + 4\theta_2\theta_1 + 4\theta_1^2 - 3\theta_1^2 - 6\theta_1\theta_2 - 3\theta_2^2}{12}$$

$$= \frac{\theta_2^2 - 2\theta_2\theta_1 + \theta_1^2}{12} = \frac{(\theta_2-\theta_1)^2}{12}$$

15] p175 The constants that determine a pdf or prob. fn are called the parameters of the distribution.

ex)

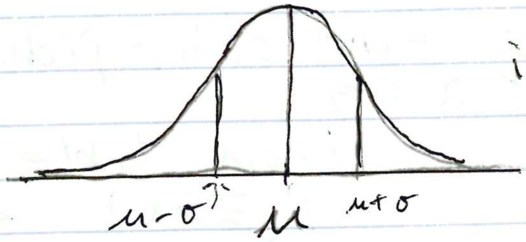
dist	parameters	
bin (n, p)	n, p	(often n is known)
poisson (λ)	λ	
uniform (θ_1, θ_2)	θ_1, θ_2	
normal (μ, σ^2)	μ, σ^2	etc

§ 4.5 16] p178 * Y has a normal distribution with parameters μ and σ if for $\sigma > 0$, $f(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$, $-\infty < y < \infty$

notation not in text

17] p178 Know If $Y \sim N(\mu, \sigma^2)$, then

$E(Y) = \mu \qquad V(Y) = \sigma^2.$



inflection points at $\mu \pm \sigma$

18] The Z-score: If Y is a RV with $EY = \mu$ and $SD(Y) = \sigma$, then the Z score

19.5

$Z = \frac{Y-\mu}{\sigma}$ has $EZ = 0$ and $V(Z) = 1$.

proof] $E(Z) = \frac{1}{\sigma} E(Y-\mu) = \frac{1}{\sigma}(\mu-\mu) = 0$

$V(Z) = EZ^2 - 0 = E\left(\frac{Y-\mu}{\sigma}\right)^2 = \frac{1}{\sigma^2} E(Y-\mu)^2$

$= \frac{\sigma^2}{\sigma^2} = 1$.

19] If Y is $N(\mu, \sigma^2)$, then

$Z = \frac{Y-\mu}{\sigma}$ is $N(0,1)$ standard normal

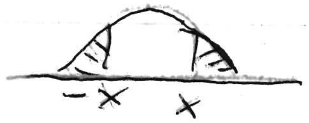
20] Know Using the normal table from the front of the book:

If $Y \sim N(\mu, \sigma^2)$ and $Z \sim N(0,1)$, then

i) $P(Y \geqq b) = P\left(Z \geqq \frac{b-\mu}{\sigma}\right)$

ii) $P(Y \leqq a) = 1 - P(Y \geqq a) = 1 - P\left(Z \geq \frac{a-\mu}{\sigma}\right)$

iii) If $w < 0$, $P(Z < w) = P(Z > -w)$



iv) If $x > 0$, $P(Z > x) = P(Z < -x)$.

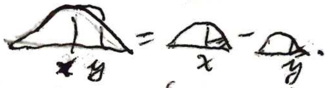
Table gives $P(Z > x)$ where $x > 0$.

Use symmetry to find other probabilities.

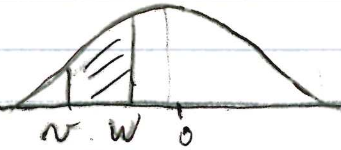
Let $v < w < 0 < x < y$

	quantity		from table
v)	$P(Z > x)$	$=$	$P(Z > x)$
vi)	$P(Z < x)$	$=$	$1 - P(Z > x)$
vii)	$P(Z < w)$	$=$	$P(Z > -w)$
viii)	$P(Z > w)$	$= P(Z < -w) =$	$1 - P(Z > -w)$

ix) $P(x<z<y) = P(z>x) - P(z>y)$



x) $P(v<z<w) = P(-w<z<-v) = P(z>-w) - P(z>-v)$



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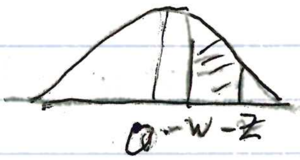

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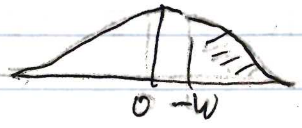

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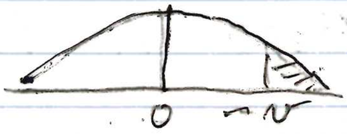
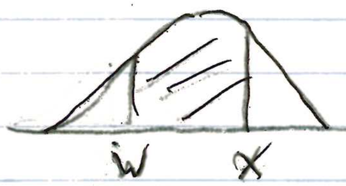


xi) $P(w<z<x) = P(z>w) - P(z>x) = 1 - P[z>-w] - P(z>x)$



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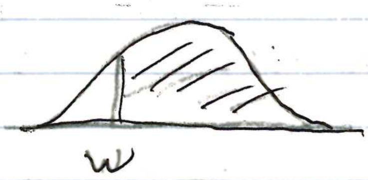

Subtract off both tails

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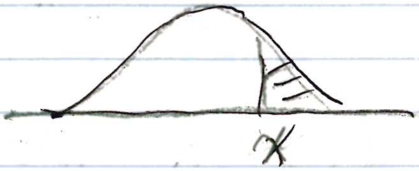
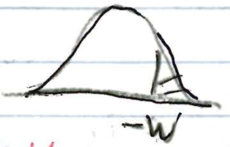


= 1 −



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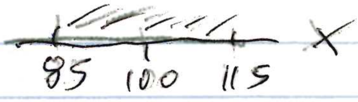
Common problem

ex] Forwards calculation   Given X or x_1 and x_2, Find a Probability.

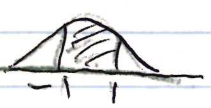
IQ's X are normal with $\mu = 100$, $\sigma = 15$.

a) $A = P(85 \leq X < 115)$ step1



step 2   z scores $\frac{85-100}{15} = -1$ $\frac{115-100}{15} = 1$

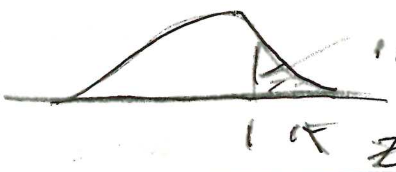
step3 $A = P(-1 < z < 1)$



step4 z table $A = 1 - P(z>1) - P(z>1) =$

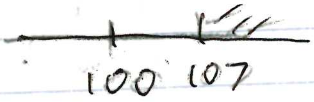
$1 - .1587 - .1587 = .6826$

z	00
1.0	$\boxed{.1587}$



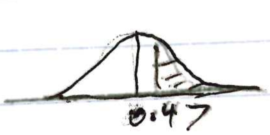
z table shows this

b) $A = P(X \geq 107)$ step 1



step 2 z score $\frac{107-100}{15} = \frac{7}{15} \approx \underbrace{0.47}_{\text{table has 2 places}}$

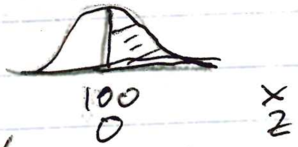
step 3 $A = P(z > 0.47)$



Step 4 table

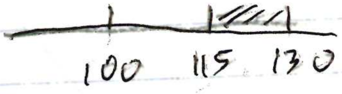
	07
.4	$\boxed{.3192}$

c) $P(X > 100) = .50$



$\frac{100-100}{15} = 0$ $P(z>0) = 1/2$

d) $P(115 < X < 130)$ step 1



step 2 z score $\frac{115-100}{15} = 1$, $\frac{130-100}{15} = 2$

step 3 $A = P(1 < z < 2)$

step 4 table $= P(z > 1) - P(z > 2) = .1587 - .0228$

$= 0.1359$

2] common problem

Backwards calculation Given a percentile α or probability, find X_α.
The α percentile of Z is the value Z_α such that $P(Z \leq Z_\alpha) = \alpha$, $0 < \alpha < 1$. (*)

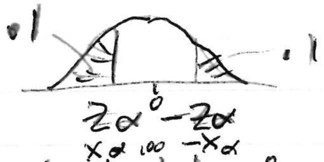
$z_\alpha = \frac{X_\alpha - \mu}{\sigma}$ so $X_\alpha = \mu + \sigma z_\alpha$.

483 21

ex] IQ scores are normal with $\mu = 100$, $\sigma = 15$. a) Find the 10th percentile of the IQ scores.

bottom 10%

step 1 picture



step 2 standard normal table: on the inside of the table, find the value closest to .1000

	.08	
1.2	.1003	.0985

^closer

so $-z_\alpha = 1.28$ and $z_\alpha = -1.28$

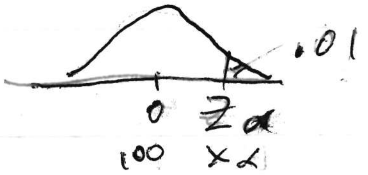
step 3 unstandardize $\left(z_\alpha = \frac{x_\alpha - \mu}{\sigma}\right)$

so $x_\alpha = \mu + \sigma z_\alpha = 100 + 15(-1.28) = \boxed{81.8}$.

b) Find the score needed to be in the top 1%

99th percentile

Step 1



Step 2

	.02	.03
2.3	.0102	.0099

closer

step 3 $z_\alpha = 2.33$

so $x_\alpha = \mu + \sigma z_\alpha = 100 + 15(2.33) = 134.99$

check: $P(X \le 134.99) = P\left(Z \le \frac{134.99 - \mu}{\sigma}\right) = P(Z \le 2.33)$

$= 1 - P(Z > 2.33) = 1 - .0099 = \boxed{.9901}$.

§4.6 22] $\overset{\text{know}}{\text{p188}}$ Y has an exponential distribution with parameter $\beta > 0$ if the pdf of Y is $f(y) = \begin{cases} \frac{1}{\beta} e^{-y/\beta}, & 0 \le y \\ 0 & \text{else} \end{cases}$.

21.5

23] p188 Know: If Y is EXP (β), then

$$E(Y) = \beta \qquad \text{and } V(Y) = \beta^2.$$

Note: p188 The lifetimes of electrical devices are often modeled by an exponential distribution.

24] p185 The gamma function is

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy, \qquad \alpha > 0.$$

25] properties of the gamma function:

i) If $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$.

ii) If n is a positive integer, $\Gamma(n) = (n-1)!$

and $\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2})\cdots\frac{1}{2}\Gamma(\frac{1}{2})$.

iii) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

proof i) Integration by parts says $\int_a^b u\,dv$

$= uv\Big]_a^b - \int_a^b v\,du$. Assume $b=\infty$ is ok for the gamma function.

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

Let $u = x^{\alpha-1}$, $dv = e^{-x}dx$. Then $du = (\alpha-1)\,x^{\alpha-2}dx$ and $v = -e^{-x}$.

$$\therefore \Gamma(\alpha) = \int_0^\infty u\,dv = [uv]_0^\infty - \int_0^\infty v\,du$$

$$= -x^{\alpha-1} e^{-x}\Big]_0^\infty + (\alpha-1)\underbrace{\int_0^\infty x^{\alpha-2} e^{-x}\, dx}_{\Gamma(\alpha-1)}$$

$$= -0 - 0 + (\alpha-1)\,\Gamma(\alpha-1)$$

($x^k e^{-x} \to 0$ as $x \to \infty$; for any k, exponential fn dominates) proof of iii) Since the $N(0,1)$ density is a pdf,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,(1)}\, e^{-\frac{1}{2}x^2}\,dx = 1$$

or $$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

or $$\int_{0}^{\infty} e^{-x^2/2}\,dx \underset{\uparrow}{=} \frac{1}{2}\sqrt{2\pi} = \sqrt{\frac{\pi}{2}}\,.$$

$e^{-x^2/2}$ is an even function

Now $\Gamma(\frac{1}{2}) = \int_0^\infty x^{-1/2} e^{-x}\,dx$.

Let $u = \sqrt{2}\sqrt{x}$ so $x = \frac{1}{2}u^2$, $dx = u\,du$

$x=0 \to u=0$, $x=\infty \to u=\infty$. so

$$\Gamma(\tfrac{1}{2}) \underset{\overset{\uparrow}{x^{-1/2} = \frac{\sqrt{2}}{u}}}{=} \sqrt{2}\int_0^\infty \frac{1}{u}\, e^{-\frac{1}{2}u^2}\, u\,du = \sqrt{2}\int_0^\infty e^{-u^2/2}\,du$$

$$= \sqrt{2}\sqrt{\frac{\pi}{2}} = \sqrt{\pi}\,.$$

26] p185 Know
Y has a gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ if

$$f(y) = \begin{cases} \dfrac{y^{\alpha-1}\, e^{-y/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)}, & 0 \le y \\ 0 & \text{else} \end{cases}$$

27] Know
p188 If Y is gamma (α, β), then

$EY = \alpha\beta$ and $V(Y) = \alpha\,\beta^2$.

28] Many random variables W are such that $g(W)$ is gamma (α,β) for an appropriate g.

P187 22.5

$\nu =$ nU

29] Y has a <u>chi squared distribution</u> with parameter ν, $Y \sim \chi^2_\nu$, if ν is a positive integer and Y is a gamma $(\alpha = \frac{\nu}{2}, \beta = 2)$ RV.

30] P188 If Y is χ^2_ν, then $EY = \nu$ and $V\,Y = 2\nu$.

31] If Y is $EXP(\beta)$, then Y is a gamma $(\alpha = 1, \beta)$ RV.

proof $f(y) = \frac{1}{\beta} e^{-y/\beta}$, $y > 0$

and if Y is gamma $(\alpha = 1, \beta)$, then

$$f(y) = \frac{y^{1-1} e^{-y/\beta}}{\beta^1 \Gamma(1)} = \frac{e^{-y/\beta}}{\beta}, \quad y > 0,$$

and the 2 pdfs are the same.

32] u substitution $\int_a^b f(g(x))\, g'(x)\, dx$

$$= \int_{g(a)}^{g(b)} f(u)\, du = \int_c^d f(u)\, du = F(d) - F(c)$$

$$= F(g(b)) - F(g(a)) \quad \text{where}$$

$u = g(x)$, $c = g(a)$ and $d = g(b)$ and $F' = f$.

ex] $$\int_a^b \frac{1}{\beta} e^{-x/\beta}\, dx = \int_a^b e^{-x/\beta} \frac{dx}{\beta} \underset{\uparrow}{=} \int_{\frac{a}{\beta} = g(a)}^{\frac{b}{\beta} = g(b)} e^{-u}\, du$$

$$\boxed{u = x/\beta = g(x),\ du = \frac{1}{\beta} dx, \quad x = a \to u = \frac{a}{\beta},\ x = b \to u = \frac{b}{\beta}}$$

$$= -e^{-u} \Big|_{\frac{a}{\beta}}^{b/\beta} = -e^{-x/\beta} \Big|_a^b = -e^{-b/\beta} + e^{-a/\beta}$$

$$= e^{-a/\beta} - e^{-b/\beta} \qquad \text{where } 0 < a < b < \infty.$$