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33] If Y is $EXP(\beta)$, then the DF of Y

is $F(y) = P(Y \le y) = \int_0^y \frac{1}{\beta} e^{-x/\beta} dx$

$F(y) = \begin{cases} 1 - e^{-y/\beta}, & \text{for } y \ge 0 \\ 0, & y \le 0 \end{cases}$ by the

previous ex with $a=0$ and $b=y$.

§4.7

34] p194 A RV Y has a beta dist with parameters $\alpha > 0$ and $\beta > 0$ if

$$f(y) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1} & 0<y<1 \\ 0 & \text{else.} \end{cases}$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

35] If Y is $\text{beta}(\alpha, \beta)$, then

$EY = \frac{\alpha}{\alpha+\beta}$ and $V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

§4.9

36] p 202 EY^k and $E(Y-\mu)^k$

are the kth and kth central moments of Y

37] p202 * $m(t) = E\, e^{tY}$ is the mgf of Y.

$= \begin{cases} \sum e^{ty} p(y), & Y \text{ discrete} \\ \int_{-\infty}^{\infty} e^{ty} f(y)\, dy, & Y \text{ continuous} \end{cases}$

38] The mgf of $g(Y)$ is $E(e^{tg(Y)}) = \begin{cases} \sum e^{tg(y)} p(y), & Y \text{ discrete} \\ \int_{-\infty}^{\infty} e^{tg(y)} f(y)\, dy, & Y \text{ contin} \end{cases}$

p205 $E(Y^k) = m^{(k)}(0)$.

Note p207 Tchebysheff's th again

If $EY = \mu$ and $V(Y) = \sigma^2$,

$P(|Y-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$, $P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

39] Not in text. Let $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_r)$ be the parameters of a probability function $p(y)$ or probability density function $f(y)$.

good technique but cover it too fast in 483

Let $p(y) = c(\underset{\sim}{\theta})\ k(y|\underset{\sim}{\theta})$, Y disc

$f(y) = \underbrace{c(\underset{\sim}{\theta})}_{\text{constant}}\ \underbrace{k(y|\underset{\sim}{\theta})}_{\text{kernel}}$, Y contin

Then $k(y|\underset{\sim}{\theta})$ is the Kernel function

Kernel means "essential part."

Then $1 = \int_{-\infty}^{\infty} c(\underset{\sim}{\theta})\ k(y|\underset{\sim}{\theta})dy$.

So $\frac{1}{c(\underset{\sim}{\theta})} = \int_{-\infty}^{\infty} k(y|\underset{\sim}{\theta})\,dy$ if Y is contin.

Similarly, $1 = \sum_y c(\underset{\sim}{\theta})\ k(y|\underset{\sim}{\theta})$.

so $\frac{1}{c(\underset{\sim}{\theta})} = \sum_y k(y|\underset{\sim}{\theta})$ if Y is disc.

The kernel function technique is useful for finding $E(g(Y))$.

Often $E\,g(Y) = \int_{-\infty}^{\infty} g(y)\ c(\underset{\sim}{\theta})\ k(y|\underset{\sim}{\theta})\,dy$

$$= a\,c(\underset{\sim}{\theta}) \underbrace{\int_{-\infty}^{\infty} k(y|\underset{\sim}{\tau})\,dy}_{1/c(\underset{\sim}{\tau})}$$

$$= a\,c(\underset{\sim}{\theta}) \frac{1}{c(\underset{\sim}{\tau})} \underbrace{\int_{-\infty}^{\infty} c(\underset{\sim}{\tau})\,k(y|\underset{\sim}{\tau})\,dy}_{1}$$

$$= a\,c(\underset{\sim}{\theta}) \frac{1}{c(\underset{\sim}{\tau})}$$ if Y is Contin.

If Y is discrete, often

$$E\,g(Y) = \Sigma\, g(y)\, c(\underset{\sim}{\theta})\, k(y|\underset{\sim}{\theta})$$

$$= a\,c(\underset{\sim}{\theta})\, \Sigma\, k(y|\underset{\sim}{\tau}) = a\,c(\underset{\sim}{\theta}) \frac{1}{c(\underset{\sim}{\tau})} = \frac{a\,c(\underset{\sim}{\theta})}{c(\underset{\sim}{\tau})} \underbrace{\Sigma\, c(\underset{\sim}{\tau})\,k(y|\underset{\sim}{\tau})}_{1}.$$

This trick is especially useful for beta, Gamma, and normal distributions.
Find EY^k if $p(y)$ or $f(y)$ has y^α in the formula.
Find Ee^{tY} if $p(y)$ or $f(y)$ has $e^{-\alpha y}$ in the formula.

ex] $$\int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \int_{-\infty}^{\infty} k_{N(0,1)}(y)\,dy = \frac{1}{c(0,1)}$$

$$= \sqrt{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}}_{N(0,1)\text{ pdf}} dy = \sqrt{2\pi}.$$

ex] $f(x) = \dfrac{\beta\,\alpha^\beta}{x^{\beta+1}}$, $x > \alpha$, $\alpha > 0, \beta > 0$.

Find EX^r with the kernel method.

Soln $\int_\alpha^\infty \beta\alpha^\beta \frac{1}{x^{\beta+1}}\,dx = 1$ so $\underset{\sim}{\theta} = (\alpha, \beta)$.

$$\frac{1}{c(\alpha,\beta)} = \int_\alpha^\infty \frac{1}{x^{\beta+1}}\,dx = \frac{1}{\beta\alpha^\beta}. \text{ Thus}$$

$$E\,x^r = \int_\alpha^\infty x^r \frac{\beta\alpha^\beta}{x^{\beta+1}}\,dx = \beta\alpha^\beta \int_\alpha^\infty \frac{1}{x^{\beta-r+1}}\,dx$$

$$\underset{\sim}{\tau} = (\alpha, \beta-r)$$

$$= \beta\alpha^\beta \frac{1}{(\beta-r)\,\alpha^{\beta-r}}$$

$$= \frac{\beta}{\beta-r}\alpha^r \text{ for } \beta > r \text{ (need } \beta-r>0).$$

$$\left(\frac{1}{c(\alpha,\beta)} = \int_\alpha^\infty \frac{1}{x^{\beta+1}}\,dx = \frac{1}{\beta\alpha^\beta} \text{ so } \frac{1}{c(\alpha,\beta-r)} = \int_\alpha^\infty \frac{1}{x^{\beta-r+1}}\,dx = \frac{1}{(\beta-r)\alpha^{\beta-r}}\right)$$

Also examine proofs for $E(Y)$ and $E[Y(Y-1)]$ for binomial and poisson distributions.

Read ex 4.16 on p205 carefully (complete the square).

p 203

ex] mgf of Gamma (α, β) RV

$$m(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} f(y)\, dy$$

$$= \int_0^{\infty} e^{ty} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)}\, dy$$

f(y) is defined on $(0,\infty)$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^{\infty} \underbrace{y^{\alpha-1} \exp\left[-y\left(\tfrac{1}{\beta}-t\right)\right]}_{\text{kernel of a Gamma}(\alpha,\eta)\text{ RV}} dy$$

where $\frac{1}{\eta} = \frac{1}{\beta} - t = \frac{1-t\beta}{\beta}$

So $\eta = \frac{\beta}{1-\beta t}$.

Now $\int_0^{\infty} y^{\alpha-1} e^{-y/\beta} dy = \frac{1}{c(\alpha,\beta)} = \beta^{\alpha}\Gamma(\alpha)$.

So $m(t) = c(\alpha,\beta)\frac{1}{c(\alpha,\eta)} = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \eta^{\alpha}\Gamma(\alpha)$

$= \left(\frac{\eta}{\beta}\right)^{\alpha} = \left(\frac{1}{1-\beta t}\right)^{\alpha}$ for $t < \frac{1}{\beta}$

since $\eta > 0$ means $\frac{1}{\beta} - t > 0$ or $t < \frac{1}{\beta}$.

Now $m'(t) = \frac{d}{dt}(1-\beta t)^{-\alpha} = -\alpha(1-\beta t)^{-\alpha-1}(-\beta)$

$= \alpha\beta(1-\beta t)^{-(\alpha+1)}$

and $m'(0) = \alpha\beta = E(Y)$.

Now $m''(t) = \frac{d}{dt} \alpha\beta (1-\beta t)^{-(\alpha+1)}$

$= -\alpha\beta(\alpha+1)(1-\beta t)^{-\alpha-1-1}(-\beta)$

$= \alpha\beta^2(\alpha+1)(1-\beta t)^{-\alpha-2}$.

So $EY^2 = \alpha\beta^2(\alpha+1)$

and $V(Y) = EY^2 - (EY)^2$

$= \alpha\beta^2(\alpha+1) - (\alpha\beta)^2$

$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$

ch 5 1] p224 multivariate prob distributions are used to describe a population of several variables. Suppose that there are n variables $Y_1, \ldots, Y_n$. Then an outcome is $(Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n) \equiv (y_1, y_2, \ldots, y_n)$.

ex) Height, weight, age and gender of SIU students

2] When $n=2$, the multivariate prob dist is called a bivariate distribution.

3] * p225 Let Y_1 and Y_2 be discrete RV's. The joint probability function of Y_1 and Y_2 is given by $p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2)$ for $y_1, y_2 \in \mathbb{R}$

$\mathbb{R} = (-\infty, \infty)$.

26

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ex] M 483 students

		0=F	1=M
	nonengl 1	5	9
Y_2	eng 0	3	9
		Y_1	

Y_1 gender F (0), M (1)

Y_2 major engineer (0), nonengineer (1)

Randomly select student

$$P(Y_1=0, Y_2=0) = P(F \cap Eng) = 3/26$$

Note: Suppose Y_1 takes values $y_{11}, \dots, y_{1k}$ and Y_2 takes on values $y_{21}, \dots, y_{2,m}$.

we could let W take on values $w_1, \dots, w_{km}$ where $w_1 = y_{11}$ and $y_{21}, \dots, w_{km} = y_{1k}$ and y_{2m}

and get a univariate discrete RV. A joint distribution is useful for getting information about different categories.

4] p225 A function $p(y_1, y_2)$ is a joint prob function of 2 discrete RVs if

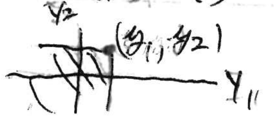
1) $p(y_1, y_2) \geq 0 \quad \forall y_1, y_2$

2) $$\sum_{(y_1,y_2):\, p(y_1,y_2)>0} p(y_1, y_2) = 1.$$

5] *p226 The joint distribution function for any two RVs Y_1 and Y_2 (discrete or continuous) is

$$F(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2), \quad y_1, y_2 \in \mathbb{R}$$

Note this is the prob of a "southwest corner:



6] p 227 Let Y_1 and Y_2 be continuous RVs with joint DF $F(y_1, y_2)$. Then $f(y_1, y_2)$ is the joint probability density function of Y_1 and Y_2 if

$$F(y_1, y_2) = \int_{-\infty}^{y_2} \int_{-\infty}^{y_1} f(t_1, t_2)\, dt_1\, dt_2$$

for $y_1, y_2 \in \mathbb{R}$.

7] If $F(y_1, y_2)$ is a joint DF, then

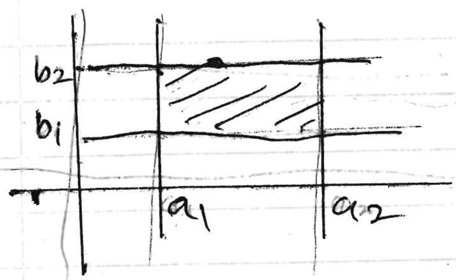
i) $F(-\infty, -\infty) = F(-\infty, y_2) = F(y_1, -\infty) = 0$

ii) $F(\infty, \infty) = 1$.

8]* p 228 The function $f(y_1, y_2)$ is a joint pdf if i) $f(y_1, y_2) \geq 0 \quad \forall\, y_1, y_2 \in \mathbb{R}$

ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1\, dy_2 = 1$

9]* $P(a_1 \leq Y_1 \leq a_2 \underset{\substack{\uparrow \\ \text{intersection}}}{,} b_1 \leq Y_2 \leq b_2) = \int_{b_1}^{b_2} \int_{a_1}^{a_2} f(y_1, y_2)\, dy_1\, dy_2$



volume under surface formed by $f(y_1, y_2)$.

see Fig 5.2

10] p 231 In the discrete case, a multivariate prob function is

$$p(y_1, \ldots, y_n) = P(Y_1 = y_1, \ldots, Y_n = y_n).$$

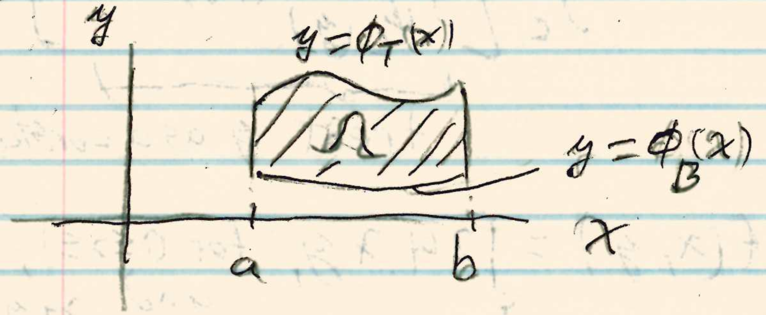
11] In the continuous case,

$$P(Y_1 \le y_1, Y_2 \le y_2, \ldots, Y_n \le y_n) = F(y_1, \ldots, y_n) =$$

$$\int_{-\infty}^{y_n} \int_{-\infty}^{y_{n-1}} \cdots \int_{-\infty}^{y_2} \int_{-\infty}^{y_1} \underbrace{f(t_1, t_2, \ldots, t_{n-1}, t_n)}_{\text{joint PDF}} \, dt_1 \cdots dt_n$$

See Exam P.5

12] Evaluation of double integrals with iterated integrals.



Ω is the region of integration

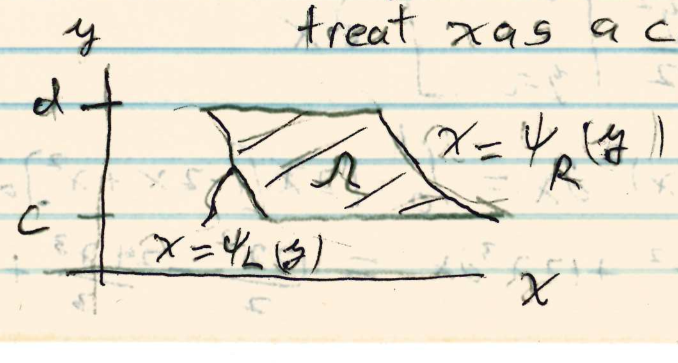
a) Suppose we want $\iint_\Omega f(x,y)\,dx\,dy = \iint_\Omega f(x,y)\,dy\,dx$

where Ω is bounded on top by the function $y = \phi_T(x)$, on the bottom by the function $y = \phi_B(x)$ and to the left and right by the lines $x = a$ and $x = b$.

Then $\iint_\Omega f(x,y)\,dx\,dy = \iint_\Omega f(x,y)\,dy\,dx$ --- match

$$= \int_a^b \Big[\underbrace{\int_{\phi_B(x)}^{\phi_T(x)} f(x,y)\,dy}_{\text{treat } x \text{ as a constant}} \Big] \, dx$$

b)



Now suppose Ω is bounded on the left by $\psi_L(y)$ and the right by $\psi_R(y)$ and on the top and bottom by $y = d$ and $y = c$. Then

match

$$\iint_\Omega f(x,y)\,dx\,dy = \int_c^d \Big[\underbrace{\int_{\psi_L(y)}^{\psi_R(y)} f(x,y)\,dx}_{\text{treat } y \text{ as a constant}} \Big] dy$$

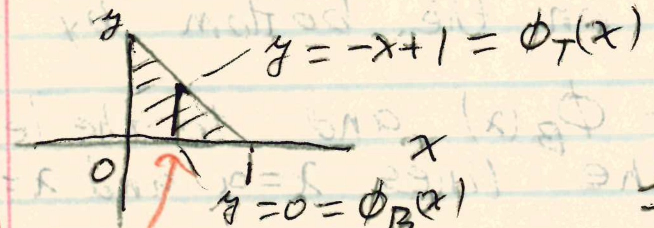
ex] Suppose $f(x,y) = \begin{cases} 24xy, & \text{for } 0 \le x \le 1,\ 0 \le y \le 1 \text{ and } x+y \le 1 \\ 0, & \text{else} \end{cases}$

set inequalities to = to get lines

Show $\iint f(x,y)\,dx\,dy = 1$ so that $f(x,y)$ is a pdf.

test points (0,0) and (1,1)

a)



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

make line parallel to what you are integrating dy ∥ y axis

$$= \int_0^1 \Big[\int_{\phi_B(x)}^{\phi_T(x)} f(x,y)\,dy \Big] dx$$

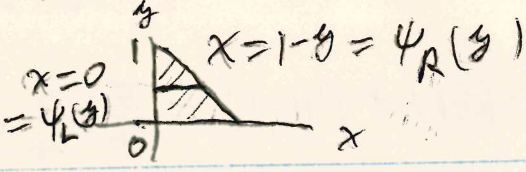
$$= \int_0^1 \Big[\int_0^{1-x} 24xy\,dy \Big] dx$$

$$= \int_0^1 \Big[24x \frac{y^2}{2} \Big|_{y=0}^{y=1-x} \Big] dx$$

$$= \int_0^1 12x(1-x)^2\,dx = \int_0^1 12x[1-2x+x^2]\,dx$$

$$= \int_0^1 12x - 24x^2 + 12x^3\,dx = \frac{12x^2}{2} - \frac{24x^3}{3} + \frac{12x^4}{4} \Big|_0^1$$

$= 6 - 8 + 3 = 1$

b) 

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

make line parallel to what you are integrating $dx \parallel$ x axis

$$= \int_0^1 \left[\int_{\psi_L(y)}^{\psi_R(y)} f(x,y)\,dx \right] dy =$$

$$\int_0^1 \left[\int_0^{1-y} 24\,xy\,dx \right] dy = \int_0^1 \left[24y \frac{x^2}{2} \Big|_{x=0}^{x=1-y} \right] dy$$

$$= \int_0^1 12\,y(1-y)^2\,dy = \int_0^1 12\,y\,(1-2y+y^2)\,dy$$

$$= \int_0^1 12y - 24y^2 + 12y^3\,dy =$$

$$\frac{12y^2}{2} - \frac{24y^3}{3} + \frac{12y^4}{4}\Big|_0^1 = 6 - 8 + 3 = 1$$

See 2nd to last paragraph on p231 and ex 5.4 on p230-231

§5.3 13 * p236 If Y_1 and Y_2 have joint prob function $p(y_1, y_2)$, then the marginal prob functions for Y_1 and Y_2 are

$$p_{Y_1}(y_1) = \sum_{y_2} p(y_1, y_2),\ \forall y_1 \text{ and } p_{Y_2}(y_2) = \sum_{y_1} p(y_1, y_2)\quad \forall y_2$$

y_1: hold fixed $\qquad$ y_2: hold fixed

ex)

			Y_1 gender F 0	M 1
major Y_2	noneng	1	$\frac{5}{26}$	$\frac{9}{26}$
	eng	0	$\frac{3}{26}$	$\frac{9}{26}$

$$p_{Y_1}(1) = P(M) = p(1,0) + p(1,1) = \frac{9}{26} + \frac{9}{26} = \frac{18}{26}$$
$$p_{Y_1}(0) = P(F) = p(0,0) + p(0,1) = \frac{3}{26} + \frac{5}{26} = \frac{8}{26}$$

28.5

$p_{Y_2}(0) = P(\text{eng}) = p(0,0) + p(1,0) = \frac{3}{26} + \frac{9}{26} = \frac{12}{26}$

$p_{Y_2}(1) = P(\text{noneng}) = p(0,1) + p(1,1) = \frac{5}{26} + \frac{9}{26} = \frac{14}{26}$.

Note The marginal prob function of Y_i is just the univariate prob function of Y_i

k	0	1
$P(Y_1 = k)$	$\frac{8}{26}$	$\frac{18}{26}$

$\leftarrow \underset{\text{probability}}{p_{Y_1}(y)} \in [0,1]$

$\sum p_{Y_1}(y) = 1$

E2 review P6

14) Know Let Y_1 and Y_2 have joint pdf $f(y_1, y_2)$. Then the marginal pdfs of Y_1 and Y_2 are

$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 \quad \forall y_1$

hold y_1 fixed

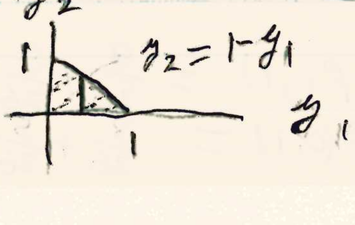
and $f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 \quad \forall y_2$.

hold y_2 fixed.

Note A marginal pdf is simply the univariate pdf.

15) Common problem: find a marginal pdf from a joint pdf or a marginal prob function from a joint prob function. See last ex.

ex] $f(y_1, y_2) = \begin{cases} 24 y_1 y_2, & 0 \le y_1 \le 1 \quad 0 \le y_2 \le 1 \\ & y_1 + y_2 \le 1 \\ 0, & \text{else} \end{cases}$



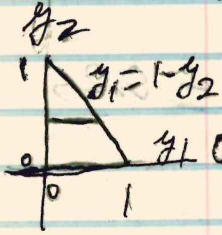
make line parallel to what you are integrating dy_2 $dy_2 \parallel y_2$ axis m483 29

$$f_{Y_1}(y_1) = \underbrace{\int_{-\infty}^{\infty}}_{\text{do match}} f(y_1, y_2)\, dy_2 = \int_0^{1-y_1} \underbrace{24\, y_1 y_2\, dy_2}_{\text{don't match}}$$

$$= 24 \frac{y_1\, y_2^2}{2} \Big|_0^{1-y_1} = 12\, y_1 (1-y_1)^2, \quad 0 \le y_1 \le 1$$

So $f_{Y_1}(y_1) = \begin{cases} 12\, y_1 (1-y_1)^2, & 0 \le y_1 \le 1 \\ 0, & \text{else.} \end{cases}$

By <u>symmetry</u> $f_{Y_2}(y_2) = \begin{cases} 12\, y_2 (1-y_2)^2, & 0 \le y_2 \le 1 \\ 0, & \text{else} \end{cases}$



or $f_{Y_2}(y_2) = \underbrace{\int_{-\infty}^{\infty}}_{\text{do match}} f(y_1, y_2)\, dy_1 = \int_0^{1-y_2} \underbrace{24\, y_1 y_2\, dy_1}_{\text{don't match}}$

$$= 24 \frac{y_1^2}{2} y_2 \Big|_0^{1-y_2} = 12 (1-y_2)^2 y_2, \quad 0 \le y_2 \le 1$$

16] * If Y_1 and Y_2 have a joint prob function given by a table, get the marginal prob's from the row and col sums

ex]

	y_2	gender y_1: F 0	M 1	row sum for Y_2: $P(Y_2 = y_2)$
eng	0	5/26	9/26	14/26
noneng	1	3/26	9/26	12/26
col sum for Y_1: $P(Y_1 = y_1)$		8/26	18/26	grand total = 1

17] * p 239 If Y_1 and Y_2 have joint prob function $p(y_1, y_2)$ and Y_2 has marginal prob function $p_{Y_2}(y_2)$, then the conditional (discrete) prob function of Y_1 given Y_2 is

$$p_{Y_1|Y_2}(y_1 | y_2) = P(Y_1 = y_1 | Y_2 = y_2) = \frac{p(y_1, y_2)}{p_{Y_2}(y_2)}$$

if $p_{Y_2}(y_2) > 0$.

29.5

Similarly $p_{Y_2|Y_1}(y_2|y_1) = \dfrac{p(y_1, y_2)}{p_{Y_1}(y_1)}$ if $p_{Y_1}(y_1) > 0$.

ex)

		F (Y_1) 0	M 1	$p_{Y_2}(y_2)$
eng	Y_2 0	3/26	9/26	12/26
noneng	1	5/26	9/26	14/26
$p_{Y_1}(y_1)$		8/26	18/26	26/26

$p_{Y_1|Y_2=0}(y_1=0 \mid y_2=0) = p_{Y_1|Y_2=0}(0|0) = P(F \mid \text{Engineer})$

$= \dfrac{p(0,0)}{p_{Y_2}(y_2=0)} = \dfrac{3/26}{12/26} = \dfrac{3}{12} = \dfrac{1}{4} = 0.25$

$p_{Y_2|Y_1=0}(y_2=1 \mid y_1=0) = \dfrac{p(0,1)}{p_{Y_1}(0)} = \dfrac{5/26}{8/26} = \dfrac{5}{8} = 0.625$

$= P(\text{nonengineer} \mid \text{female})$

18) Know p 241. Let Y_1 and Y_2 have joint pdf $f(y_1, y_2)$ and marginal pdf's $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$.

Then the conditional pdf of Y_1 given $Y_2 = y_2$

is $f_{Y_1|Y_2=y_2}(y_1|y_2) = \dfrac{f(y_1, y_2)}{f_{Y_2}(y_2)}$ for any $y_2 \ni$ (such that) $f_{Y_2}(y_2) > 0$

(y_2 fixed)

and the conditional pdf of Y_2 given $Y_1 = y_1$

is $f_{Y_2|Y_1=y_1}(y_2|y_1) = \dfrac{f(y_1, y_2)}{f_{Y_1}(y_1)}$ for any $y_1 \ni$ $f_{Y_1}(y_1) > 0$.

(y_1 fixed)